Lectures on Numerical Methods

— How to Solve/Use Life Cycle Models —

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Introduction

- Basic framework (want to solve):
  - Life cycle versions of Bewley-Aiyagari-Huggett model
- Features of the model
  1. Idiosyncratic risks ⇒ heterogeneity
  2. Aggregation ⇔ distribution dynamics
     - Steady state, transition, and dynamics

Numerical Methods

- *Global* solution methods
  - ⇔local approximation methods (e.g., linear quadratic approximations)
    - around the deterministic steady-state
    - RBC, New Keynesian DSGE models etc.
    - called "Perturbation method"
  - want to know *policy functions*
- Why global solution methods?
  1. You may not know steady states before solving the problem
  2. Heterogeneous agents model: super rich and poor
  3. Income risks that individuals face is usually very large
     - ⇒Policy functions are potentially nonlinear

Numerical Methods (cont.)

- Need some numerical methods (off-the-shelf techniques)
  - Optimization: Judd (1998), Chap.4
  - Nonlinear equations: Judd (1998), Chap.5
  - Functional approximation: Judd (1998), Chap.6
  - Numerical integration/differentiation: Judd (1998), Chap.7
- Useful books
  - Judd (1998), Marimon and Scott (1999), Miranda and Fackler (2002), Heer and Maussner (2009)

Numerical Methods (cont.)

1. Value function iteration (VFI)
   - Finite iteration in life cycle models
2. Endogenous gridpoint method
   - use the Euler equation

- Both methods are almost identical when solving life cycle models
- All codes used here are available from
  - http://homepage2.nifty.com/~tyamada/teaching/numerical.html
  - written in Fortran 90/95

Software Choice

- Matlab/Gauss/Scilab/Octave
  - Languages for scientific computing; matrix-oriented
  - Some useful tools: DYNARE, CompEcon (Miranda and Fackler,2002)
- C/C++/Fortran:
  - Packages (Not free): IMSL/NAG
  - Numerical recipes (Book): Press et al. (2007)
  - Subroutine libraries (Free): LAPACK/BLAS/MINPACK etc.
- Mathmatica/Maple:
  - Symbolic math

Guide Map

1. Two-period model
2. Three-period model
3. (Full) Life cycle models:
   3.1 Steady State
   3.2 Transition: Yamada (2011,JEDC)
   3.3 Aggregate shock

Two-period Model

Consider a two-period model (no uncertainty)

- Basic setup:

$$\max_{c_Y, c_O, a'} \frac{c_Y^{1-\gamma}}{1-\gamma} + \beta \frac{c_O^{1-\gamma}}{1-\gamma},$$

$$\text{s.t.}$$

$$c_Y + a' = y + a,$$

$$c_O = ss + Ra'$$

  - (c_Y, c_O): consumption, β: discount factor
  - a': savings, a: initial asset (given), R: gross interest rate
  - y: labor income (deterministic), ss: social security benefit

Calibration

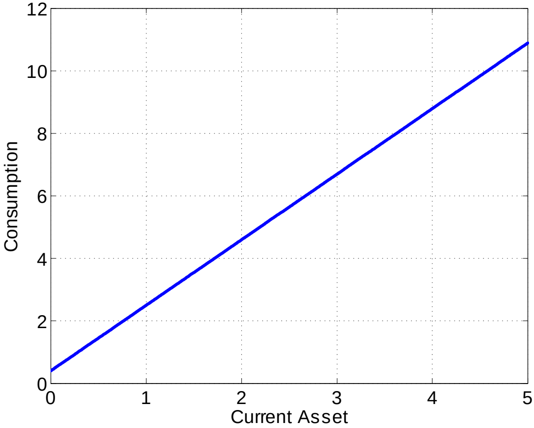
- How to solve the two-period model?
  - Backward induction
- Before solving the model quantitatively, we need parameters
  - Usually this process is called "calibration"
  - One period is 30 years: Song et al. (2009)
  - $\beta = 0.985^{30}$, $\gamma = 2$, $R = 1.025^{30}$, $y = 1$, $ss = 0.4$
- What we want to know are *policy functions*
  - $c_Y = f^Y(a)$: consumption function of young agents
  - $c_O = f^O(a')$: consumption function of old agents
  - $a' = g(a)$: saving function

Numerical Methods

1. In the last period (old), agents consume all wealth
   - $c_O = f^O(a') = ss + Ra'$
   - Note that R and ss are given
2. Given the consumption function of old household $f^O(a')$, we compute young's policy function from the Euler equation:

$$\begin{aligned} u'(c_Y) &= \beta R u'(c_O), \\ u'(y + a - a') &= \beta R u'(f^O(a')) \\ &= \beta R u'(ss + Ra') \end{aligned}$$

Consumption Function (Old):



Numerical Methods (cont.)

- How to find the policy functions $f^Y(a)$ and $g(a)$ for young households from the Euler equation?
- There are some approaches
  1. Discrete state space method
     - Solve the Euler equation over discretized asset grids
  2. Projection method: Judd (1992)
     - Approximate policy functions by polynomial
     - Finite element method: McGrattan (1996,JEDC)
  3. Endogenous gridpoint method (EGM): Carroll (2006)
  4. Parametric expectations algorithm (PEA): Christiano and Fisher (2000)

Discrete State Space Method

- Discretize the initial asset (grid): $a_i \in \{a_{\min}, \dots, a_{\max}\}$
  - **Tips**: more grids near zero (if a borrowing constraint exists)
- Solve the Euler equation for each a_i

$$u'(y + a_i - a') = \beta R u'(f^O(a')),$$

- How to solve the Euler equation?

$$u'(\underbrace{y + a_i}_{\text{given}} - \underbrace{a'}_{\text{choice}}) = \underbrace{\beta R}_{\text{params.}} u'(\underbrace{f^O(a')}_{\text{known}})$$

Discrete State Space Method (cont.)

- Use nonlinear equation solver
  - Find a *zero* of the residual:

$$\Phi(a_i) = \beta R \frac{u'(f^O(a'))}{u'(y + a_i - a')} - 1$$

- Useful techniques to find a zero:
  - Bisection method: bisect a zero between $a_{\min}$ and $a_{\max}$
  - Newton methods
  - Broyden's method: a variant of Newton methods
- You can find several *root-finding* subroutines!
  - `fzero`: Matlab

Discrete State Space Method (cont.)

- You have a combination of current asset and savings
  - $\{a_i, a_i'\} \Rightarrow \{c_i\}$
  - Use interpolation if you want to know consumption between the discretized grids, $a_i < a < a_{i+1}$

Projection Method

- In the discrete state space method, we compute a set of *discretized* asset and consumption
- Approximating the saving function over state space
  - E.g., monomial approximation

$$a' = \hat{g}(a;\phi) = \sum_{n=1}^{N} \phi_n a^{n-1}$$

- Generally Chebyshev polynomial has some useful properties

$$a' = \hat{g}(a;\phi) = \sum_{n=1}^{N} \phi_n T_n(a)$$

  - $T_n(a)$: Basis function, e.g., $\cos((n-1)\arccos a)$

Projection Method (cont.)

- Find coefficients $\{\phi_n\}_{n=1}^{N}$ that minimizes the residual over state

$$\int \Phi(a;\phi)da = 0$$

- How to compute the residual over state space?
- E.g., Collocation method
  - Given evaluation points $\{a_m\}$,

$$\Phi(a_m;\phi) = 0, \;\; m = 1, \ldots, M$$

  - $m < n$: impossible to determine coefficients $\{\phi_n\}_{n=0}^{N}$!
- Other way to evaluate the residual, see text books

Endogenous Gridpoint Method

- Usually root-findings (using nonlinear equation solver) need computation time
  - many iteration to find a zero
- Carroll (2006,EL): Endogenous Gridpoint Method
  - Change the timing of discretization and state space
    1. Discretize *next period's asset*: $a'_j \in \{a'_1, \ldots, a'_J\}$
    2. Solve consumption function over cash-on-hand

Endogenous Gridpoint Method (cont.)

- Define RHS of the Euler equation as

$$\Gamma(a'_j) \equiv \beta R u'(ss + Ra'_j)$$

- Because the marginal utility of CRRA utility function is invertible,

$$\begin{aligned} u'(c_{Y,j}) &= \Gamma(a'_j), \\ c_{Y,j}^{-\gamma} &= \Gamma(a'_j), \\ c_{Y,j} &= \Gamma(a'_j)^{-\frac{1}{\gamma}} \end{aligned}$$

Endogenous Gridpoint Method (cont.)

- We have a pair of $\{c_{Y,j}, a'_j\}$
  - $c_{Y,j} + a'_j \equiv x_j (= y + a)$ is cash-on-hand
  - We know consumption function over cash-on-hand

$$c_Y = \hat{f}^Y(x)$$

Endogenous Gridpoint Method (cont.)

But, what we really want to know is consumption over current asset

- Retrieve current asset from the cash-on-hand

$$
\begin{aligned}
x_j &= y + a_j, \\
a_j &= x_j - y
\end{aligned}
$$

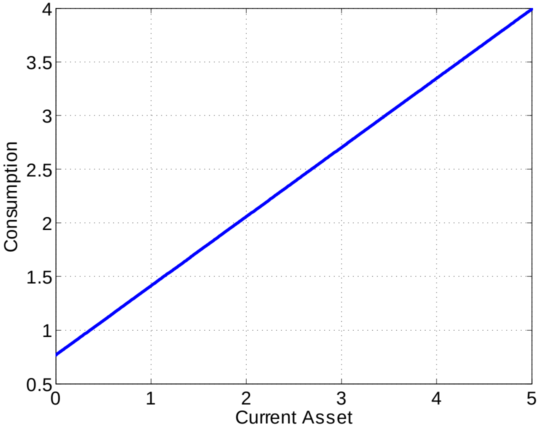
- Policy function is a pair of $\{c_j, a_j\}$: $c_j = \tilde{f}^Y(a_j)$, for $j = 1, \ldots, J$

Numerical Examples

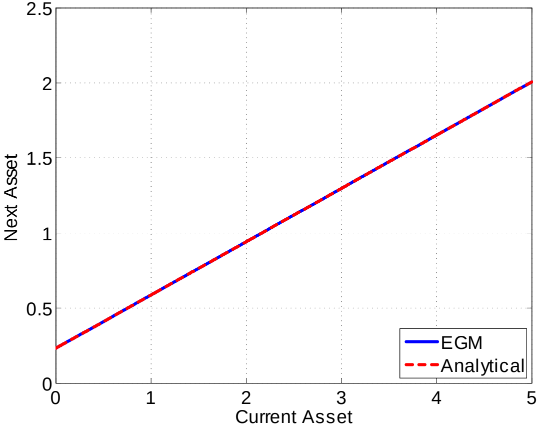
1. Consumption function when old
   - mentioned above
2. Consumption function when young
3. Saving function
4. Numerical errors
   - Closed-form solution

$$a' = \frac{y + a - [\beta R]^{-\frac{1}{\gamma}} ss}{1 + [\beta R]^{-\frac{1}{\gamma}} R}$$

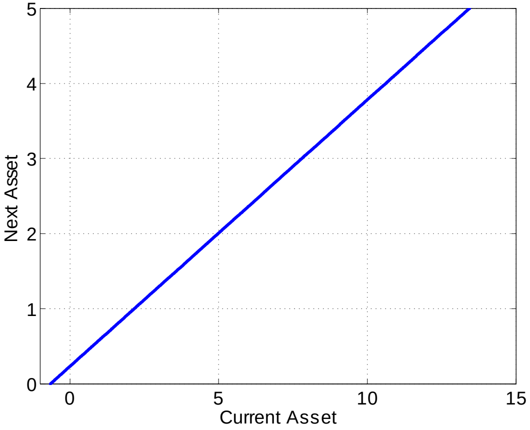
Consumption Function (Young)



Saving Function



Retrieve Current Asset



Endogenous Gridpoint Method (cont.)

- This idea is applicable to any finite (and in fact infinite) horizon problems
  - Need FOCs
- Let's apply the EGM to three period model!

Three-period model

Extend the model to three-period with uncertainty

- Consider a three-period model:

$$\max \mathbb{E}\left[\frac{c_Y^{1-\gamma}}{1-\gamma} + \beta \frac{c_M^{1-\gamma}}{1-\gamma} + \beta^2 \frac{c_O^{1-\gamma}}{1-\gamma}\right],$$

s.t.

$$c_Y + a_M = y_Y + a_Y,$$

$$c_M + a_O = y_M + Ra_M,$$

$$c_O = ss + Ra_O$$

- (c_Y, c_M, c_O): consumption, (a_Y, a_M, a_O): asset holdings
- (y_Y, y_M): labor income
- R: gross interest rate, ss: social security benefit

Three-period model (cont.)

Labor income is uncertain at middle

- E.g., labor income distribution

$$
\begin{aligned}
y_M &= \bar{y}_M + \varepsilon, \\
\varepsilon &\sim N(0, \sigma_\varepsilon^2)
\end{aligned}
$$

- Need numerical integration techniques
  - Discretize ε
  - Gauss-Legendre, Gauss-Chebyshev quadrature etc.
- Consider a very simple case: $\{y_M^{\text{high}}, y_M^{\text{low}}\}$ with prob. $\frac{1}{2}$

Three-period model (cont.)

- How to solve the model?
  - Backward induction again
- Why the three-period model is NOT a trivial extension of two-period model?
  - Need an additional state variable: y_M
  - Need functional approximation

Three-period model (cont.)

1. Old: $c_O = f^O(a_O) = ss + Ra_O$
2. Middle: solve the Euler equation for each $(a_{M,i}, y_M)$

$$u'(y_M + Ra_{M,i} - a_O) = \beta R u'(f^O(a_O))$$

   - Get a policy function $\tilde{f}^M(a_M, y_M)$ using the EGM (or other methods): $\{c_{M,i}^{\text{high}}, a_{M,i}^{\text{high}}\}$ and $\{c_{M,i}^{\text{low}}, a_{M,i}^{\text{low}}\}$
   - This step is completely the same as in the two-period model

3. Young: solve the Euler equation again

$$u'(y_Y + Ra_{Y,i} - a_M) = \beta R \mathbb{E} u'(\tilde{f}^M(a_M, y_M))$$

Three-period model (cont.)

- Policy function at middle $\tilde{f}^M(a_M, y_M)$ is a set of discretized points
  - $\{c_{M,i}^{high}, a_{Y,i}^{high}\}$, $\{c_{M,i}^{low}, a_{Y,i}^{low}\}$
- What if the choice variable a_M is between asset grids?

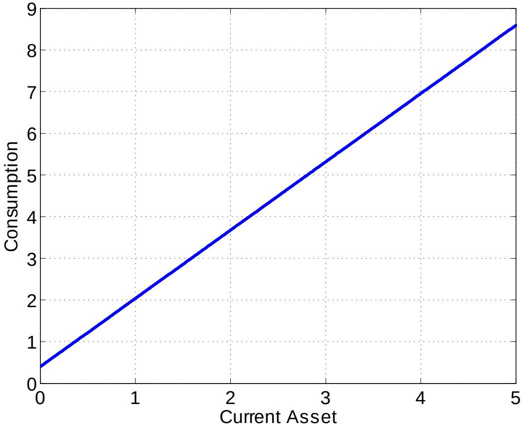
$$a_{M,i} < a_M < a_{M,i+1}$$

- Interpolation
  - Linear approximation: non-differentiable, `interp1`
  - Cubic spline interpolation: differentiable, `spline`
  - Shape-preserving spline interpolation: differentiable with concavity

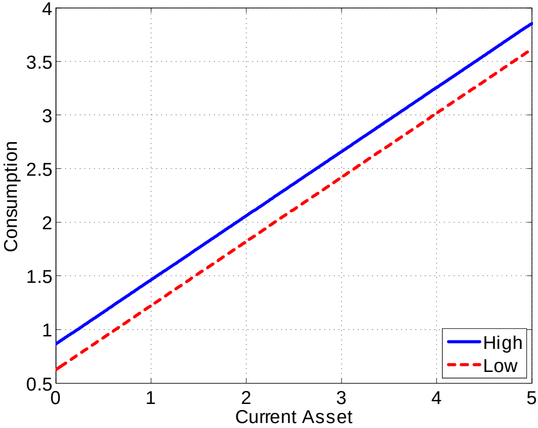
Three-period model (cont.)

- Calibration
  - One period is 20 years
  - $\beta = 0.985^{20}$, $\gamma = 2$, $R = 1.025^{20}$
  - $y_Y = 1$
  - $y_M^{\text{high}} = 1 + \epsilon$, $y_M^{\text{low}} = 1 - \epsilon$, $\epsilon = 0.2$
  - $ss = 0.4$

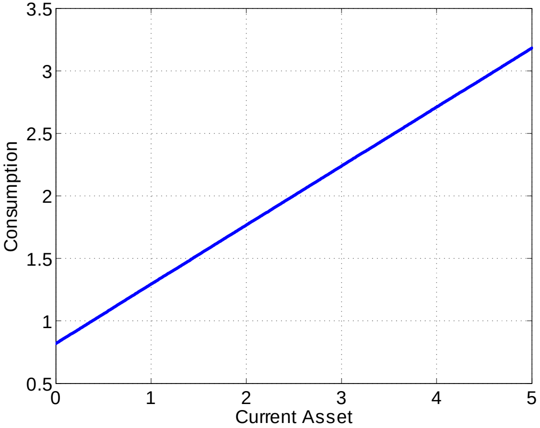
Consumption Function (Old)



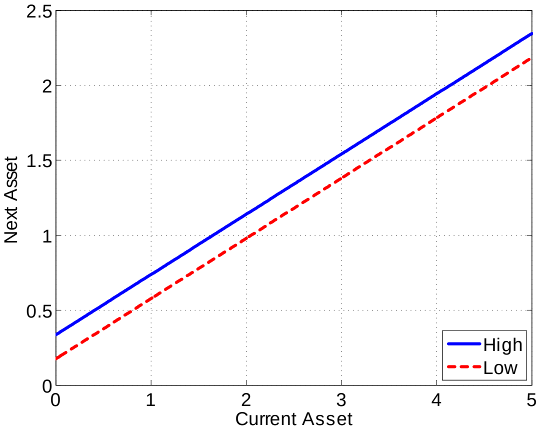
Consumption Function (Middle)



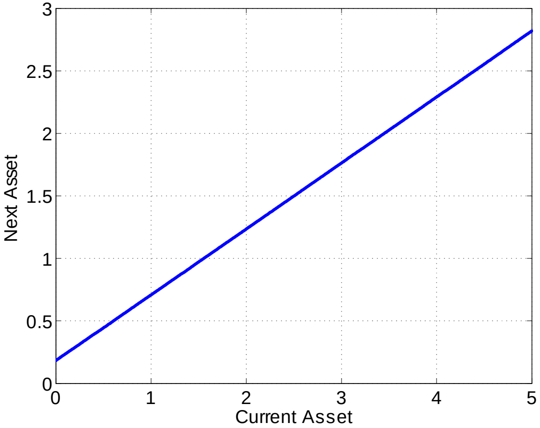
Consumption Function (Young)



Saving Function (Middle)



Saving Function (Young)



Life Cycle Models

- Generalize the simple two (three)-period models
  - Bewley/Huggett/Aiyagari framework
  - Agents live long periods
- Features of the model:
  - Life cycle – worker and retiree
  - Idiosyncratic labor income risks
  - Mortality risks (for demographic change)
  - *Dynamic general equilibrium*
    1. Steady state
    2. Transition
    3. Aggregate shocks

Household Problem

- A continuum of households exists
- There is no aggregate uncertainty
- Preferences are represented by

$$\max \mathbb{E}_1 \sum_{j=1}^{J} \xi_j \beta^{j-1} \frac{c_j^{1-\gamma}}{1-\gamma}$$

  - $j \in \{1, \dots, j^{\text{ret}}, \dots, J\}$: age
  - $\xi_j \equiv \prod_{i=1}^{j-1} \phi_i$: unconditional probability of surviving to age j

Budget Constraint

- Budget constraints for worker and retiree:

$$
\begin{aligned}
c_j + a_{j+1} &\leq (1+r)(a_j + b) + (1-\tau^{ss}) w \eta_j z, \\
c_j + a_{j+1} &\leq (1+r)(a_j + b) + ss, \\
a_{j+1} &\geq 0.
\end{aligned}
$$

  - r: interest rate, w: wage, b: accidental bequest (defined later)
  - η_j: age-specific productivity, z: idiosyncratic labor income risk
  - τ^{ss}: payroll tax for social security

Household Problem (cont.)

- Idiosyncratic labor income risks
    - Storesletten et al. (2004,JME) etc.
- Logarithms of hours worked follows

$$\ln z_{j+1} = \rho \ln z_j + \kappa_j, \ \kappa \sim N(0, \sigma_\kappa^2)$$

    - ρ: persistence, κ_j: disturbance

Household Problem (cont.)

Bellman equation for workers: $j = 1, \ldots, j^{\text{ret}}$

$$
\begin{aligned}
V_j(a,z) &= \max\left\{u(c_j) + \phi_j \beta \mathbb{E} V_{j+1}(a',z')\right\}, \\
&\quad \text{s.t.} \\
c_j + a_{j+1} &\leq (1+r)(a_j + b) + w\eta_j z, \\
a_{j+1} &\geq 0.
\end{aligned}
$$

Household Problem (cont.)

Bellman equation for retirees: $j = j^{\text{ret}} + 1, \ldots, J$

$$\begin{aligned} V_j(a) &= \max\left\{u(c_j) + \phi_j \beta V_{j+1}(a')\right\}, \\ &\text{s.t.} \\ c_j + a_{j+1} &\leq (1+r)(a_j + b) + ss, \\ a_{j+1} &\geq 0. \end{aligned}$$

First Order Conditions

- Euler equation:

$$u'\left(c_j\right) \geq \phi_j \beta \left(1+r\right) \mathbb{E} u'\left(c_{j+1}\right)$$

- Why inequality?
  - A liquidity constraint exists: $a_{t+1} \geq 0$
- What we want to know: policy function $g_j\left(a, z\right)$

Transition Law of Motion

- Probability space: $((\mathcal{A} \times \mathcal{Z}), \mathcal{B}(\mathcal{A} \times \mathcal{Z}), \Phi_j)$
  - $\mathcal{B}(\mathcal{A} \times \mathcal{Z})$: Borel σ-field
  - $\Phi_j(a, z)$: probability measure
- Transition function from current state (a, z) to next state $X \in \mathcal{B}(\mathcal{A} \times \mathcal{Z})$

$$Q_j\left((\mathcal{A} \times \mathcal{Z}), X\right) = \sum_{z' \in \mathcal{Z}} \begin{cases} \Pr\left(z, z'\right) & \text{if } g_j\left(a, z\right) \in X \\ 0 & \text{else} \end{cases}$$

- The distribution function by age:

$$\Phi_{j+1}\left(X\right) = \int Q_j\left((\mathcal{A} \times \mathcal{Z}), X\right) d\Phi_j, \quad (\forall X \in \mathcal{B}(\mathcal{A} \times \mathcal{Z}))$$

Demography

- Some households die with probability ϕ_j
- Transition of the fraction of cohort

$$\mu_{t+1} = \frac{1}{1+g}\phi_j \mu_j$$

  - μ_j: a fraction of age j, g: population growth rate
  - $\sum_{j=0}^{J} \mu_j = 1$: total population is normalized to one

Production

- Aggregate capital:

$$K = \sum_{j=1}^{J} \mu_j \int a d\Phi_j(a, z)$$

- Aggregate labor (exogenously fixed):

$$L = \sum_{j=1}^{j^{\text{ret}}} \mu_j \int \eta_j z d\Phi_j(a, z)$$

- A representative firm: Cobb-Douglas production function

$$Y = AK^{\theta} L^{1-\theta}$$

Government

- Social security system

$$\sum_{j=1}^{j^{\text{ret}}} \mu_j \int \tau^{ss} w \eta_j z_j d\Phi_j(a,z) = \sum_{j=j^{\text{ret}}+1}^{J} \mu_j ss$$
$$= \sum_{j=j^{\text{ret}}+1}^{J} \mu_j \varphi w L$$

  - $ss \overset{\text{def}}{=} \varphi w L$

- Accidental bequest

$$b = \sum_{j=1}^{J} \mu_j \int (1-\phi_j) g_j(a,z) d\Phi_j(a,z)$$

Definition of RCE

Recursive competitive equilibrium is a set of value function V, policy function g, interest rate r, wage w, tax rate τ^{ss}, and a distribution function Φ that satisfies the following conditions:

1. Household's optimality
2. Firm's optimality

$$r = \theta A K^{\theta-1} L^{1-\theta}, \; w = (1-\theta) A K^{\theta} L^{-\theta}$$

3. Market clearing conditions
   - goods, capital and labor markets
4. Government budget constraint
5. Stationarity of distribution

Steady State

Computing a Steady State: Algorithm

1. Preamble: compute aggregate labor supply L, the tax rate for social security τ^{ss}, and approximate idiosyncratic shocks
2. Initial guess: r_0
3. Solve a household's problem and get policy functions: $g_j(a, z)$
4. Compute a cumulative density function: $\Phi_j(a, z)$
5. Using the cumulative density function, compute aggregate capital K_1 and new interest rate r_1
6. Check whether new interest rate r_1 is *sufficiently* close to r_0
   6.1 Yes: It's a steady state!
   6.2 No: repeat steps 3–6 with a new interest rate

How to Solve Life Cycle Models

1. Solving household's problem
   1.1 Value function iteration (VFI)
   1.2 Projection method
   1.3 Endogenous gridpoint method (EGM)
   - This step is also applicable for structural estimation: See Gourinchas and Parker (2002), Kaplan (2010)
2. Computing density function
   2.1 Simulation
   2.2 Approximate density function
3. Find an equilibrium price (interest rate)
   - Bisection method etc.

Dynamic Programming Approach

- Good points of VFI
  - Safe (contraction mapping property): not important in life cycle models
  - Useful for nonlinear problems
  - Many application
- Bad points of VFI
  - Generally slow (but the number of iteration is fixed in life cycle models)

Endogenous Gridpoint Method

- Good points of EGM
  - Reliable
  - Fast (need no optimization)
- Bad points of EGM
  - Without FOCs, it may not be applicable (e.g., nonlinear problems)

VFI

- Basic idea is the same as in infinite horizon models
- Points
  - Find a maximum: Optimization
  - Approximation: Value function is concave
- Discretized grid

$$a_i \in \{a_{\min}, \ldots, a_{\max}\}$$

VFI (cont.)

General idea: use backward induction again!

- Age J:

$$\tilde{V}_J(a_i) = u((1+r)(a_i+b)+ss)$$

- Age $J-1$:

$$\tilde{V}_{J-1}(a_i) = \max_{a'}\{u((1+r)(a_i+b)+ss-a') + \phi_j\beta\tilde{V}_J(a')\}$$

- Iterate to age 1:

$$\tilde{V}_J(a) \Rightarrow \tilde{V}_{J-1}(a) \Rightarrow \cdots\cdots \Rightarrow \tilde{V}_{j^{ret}}(a,z) \Rightarrow \cdots\cdots \Rightarrow \tilde{V}_1(a,z)$$

VFI (cont.)

- Household's problem

$$\tilde{V}_j(a_i, z) = \max_{a'}\{u((1+r)(a_i+b)+w\eta_j z - a') + \phi_j \beta \mathbb{E}\tilde{V}_{j+1}(a', z')\}$$

- Optimization tools: `fminsearch` in Matlab
  - Golden search(≈Bisection method)
  - Quasi-Newton method
- Functional approximation:
  - $\tilde{V}_{j+1}(a', z')$ is generally strictly concave function

Endogenous Gridpoint Method (cont.)

- First order condition is defined as follows:

$$u'(c_j) \geq \phi_j \beta (1 + r) \mathbb{E} u'(c_{j+1})$$

- Discretized grid: $\tilde{a}' \in \{a_{\min}, \ldots, a_{\max}\}$, $a_{\min} = 0$
  - For example, $\#\text{grid} = 100$

Endogenous Gridpoint Method (cont.)

- Cash-on-hand of age $j+1$:

$$x' \equiv \begin{cases} (1+r)(\tilde{a}'_{j+1}+b) + w\eta_{j+1}z \\ (1+r)(\tilde{a}'_{j+1}+b) + ss \end{cases}$$

- Right hand side of the Euler equation

$$\begin{aligned} \Gamma'(\tilde{a}', z, j) &\equiv (1+r)\phi_j\beta\mathbb{E}c_{j+1}^{-\gamma} \\ &= (1+r)\phi_j\beta\mathbb{E}f_{j+1}(x', z')^{-\gamma} \end{aligned}$$

  - $c_j = \hat{f}_j(x, z)$: consumption function over cash-on-hand

Endogenous Gridpoint Method (cont.)

- FOC is redefined as

$$u'(c_j) \geq \Gamma'(\tilde{a}', z, j)$$

- Thus, we have consumption as

$$c_j = u'^{-1}\Gamma'(\tilde{a}', z, j)$$

- If the Euler equation holds with strict inequality, $a' = 0$, i.e., hand-to-mouth consumer

Tauchen's Methods

- How to calibrate idiosyncratic income risks?
  - Empirical studies: Blundell et al. (2008,AER), Storesletten et al. (2004,JPE) etc.
- How to approximate it?
  - Tauchen (1986)/Tauchen and Hussey (1992)
  - Floden (2007): approximation error of Tauchen's methods

Tauchen's Methods (cont.)

- Consider AR(1) Process
  - $y_t \equiv \log z_t$

$$y_{t+1} = \rho y_t + \kappa_t, \ \kappa \sim \mathcal{N}(0, \sigma_\kappa^2)$$

$$y_{t+1} = \rho y_t + \sigma_y (1 - \lambda^2)^{\frac{1}{2}} \tilde{\kappa}, \ \tilde{\kappa} \sim \mathcal{N}(0, 1)$$

Tauchen's Methods (cont.)

- Approximate AR(1) process by finite Markov chain
  - For example, 7-states
- Suppose $\bar{y}^N = 3\sigma_y$, $\bar{y}^1 = -3\sigma_y$: State space $Y^{\log} = \{-3\sigma_y, -2\sigma_y, -\sigma_y, 0, \sigma_y, 2\sigma_y, 3\sigma_y\}$
- Define intervals of the seven-states as follows:

$$I_1 = [3\sigma_y, \frac{5}{2}\sigma_y), I_2 = [-\frac{5}{2}\sigma_y, -\frac{3}{2}\sigma_y),$$
$$I_3 = [-\frac{3}{2}\sigma_y, -\frac{1}{2}\sigma_y), I_4 = [-\frac{1}{2}\sigma_y, \frac{1}{2}\sigma_y),$$
$$I_5 = [\frac{1}{2}\sigma_y, \frac{3}{2}\sigma_y), I_6 = [\frac{3}{2}\sigma_y, \frac{5}{2}\sigma_y),$$
$$I_7 = [\frac{5}{2}\sigma_y, 3\sigma_y)$$

Tauchen's Methods (cont.)

- Current state: $\bar{y}^i = \log z \in Y^{\log}$
  $\Rightarrow$Next state: $\bar{y}^j = \log z' \in Y^{\log}$

$$\pi_{ij} = \Pr\left(\log z' = \bar{y}^j \middle| \log z = \bar{y}^i\right) = \int_{I_j} \frac{1}{\sqrt{2\pi}\sigma_\varepsilon} e^{-\frac{1}{2}\frac{(x-\log \bar{z})^2}{\sigma_\varepsilon}} dx$$

- Range between states is defined as $w = \bar{y}^k - \bar{y}^{k-1}$
- For each i, if $j \in [2, N-1]$, then

$$\begin{aligned}\pi_{ij} &= \Pr\left[\bar{y}^j - \frac{w}{2} \le \bar{y}^j \le \bar{y}^j + \frac{w}{2}\right] \\ &= \Pr\left[\bar{y}^j - \frac{w}{2} \le \lambda \bar{y}^i + \varepsilon_t \le \bar{y}^j + \frac{w}{2}\right] \\ &= F\left(\frac{\bar{y}^j - \lambda \bar{y}^i + \frac{w}{2}}{\sigma_\varepsilon}\right) - F\left(\frac{\bar{y}^j - \lambda \bar{y}^i - \frac{w}{2}}{\sigma_\varepsilon}\right)\end{aligned}$$

Tauchen's Methods (cont.)

- As a last step, take exponent to the log AR(1) process: $\{\log z_t\} \Rightarrow \{z_t\}$
- Normalize $Ez = \sum_{z \in \mathcal{Z}} z \pi_\infty(z) = 1$ (not necessary)

$$
\begin{aligned}
\mathcal{Z} &= \{z_1, \ldots, z_7\} \\
&= \left\{ \frac{e^{-3\sigma_y}}{Ez}, \frac{e^{-2\sigma_y}}{Ez}, \frac{e^{-\sigma_y}}{Ez}, \frac{1}{Ez}, \frac{e^{\sigma_y}}{Ez}, \frac{e^{2\sigma_y}}{Ez}, \frac{e^{3\sigma_\varepsilon}}{Ez} \right\}
\end{aligned}
$$

Density Functions

- How to compute the density function?
  1. Simulation
     - Easy to implement, but errors may be large (if #sample is insufficient)
     - Easy to compute some statistics: variances, Gini etc.
  2. Approximate density function
     - Heer and Maussner (2009)/Young (2010,JEDC)

Density Functions (cont.)

Simulation-based method

1. Generate a series of idiosyncratic income shocks for S households, e.g. $S = 10,000$
   - $\{z_j^i\}_{i=1}^{10,000}$: index i represents household
2. Guess initial asset distribution a_1^i: $a_1 = 0$ by assumption
3. Using policy functions (we already get), compute a series of asset holdings

$$a_2^i = \tilde{g}_1(a_1^i, z_1^i) \rightarrow a_3^i = \tilde{g}_1(a_2^i, z_2^i) \rightarrow \cdots$$

   - Notice that some households may die before J
4. Aggregation

$$K = \sum_{j=1}^{J} \mu_j \sum_{i=1}^{10,000} a_j^i / S$$

Density Functions (cont.)

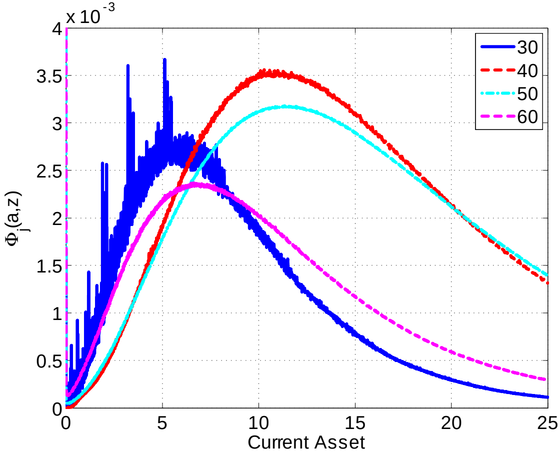
- Approximate the density function $d\Phi_j(a, z)$ by linear interpolation
  - Solve the density function forwardly
  - Take discretized grid: $a_k \in \{a_{\min}, \ldots, a_{\max}\}$
    - should be finer than policy function iteration: e.g., 10,000 grids
- A household with (a, z) saves $\hat{a}' = \tilde{g}_j(a_k, z)$
  - However, $\hat{a}'$ may not be on the discretized grid
  - Define a weight ω as follows

$$\omega = \frac{\hat{a}' - a_\ell}{a_h - a_\ell},\ \hat{a}' \in [a_\ell, a_h]$$

- Households with (a_k, z) are divided a_ℓ and a_h by the rule

$$\begin{cases} \Pr(z, z')(1-\omega) d\Phi_j(a, z) \\ \Pr(z, z')\omega d\Phi_j(a, z) \end{cases}$$

Approximated Density Function



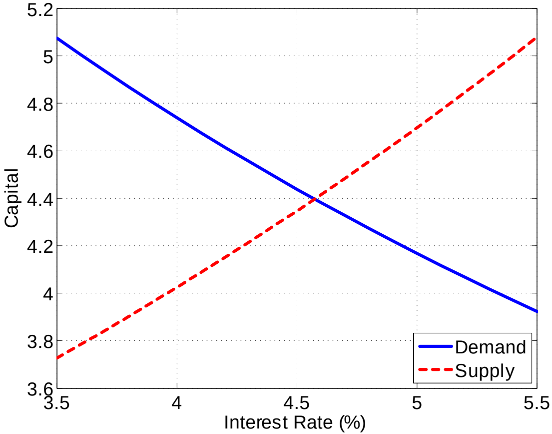
Find a Steady State

- Need to find a fixed point of interest rate
  - same as in the Bewley model with infinitely-lived agents
  - Use a bisection method: Aiyagari (1994)
- Set an initial interest rate r_0
  - update to r_1
  - if iteration error is sufficiently small, e.g. $\varepsilon = 0.00001$, stop

$$\left\| r_{k+1} - r_k \right\| < \varepsilon, \text{ or, } \left\| K^{\text{supply}} - K^{\text{demand}} \right\| < \epsilon$$

- Aggregate demand and supply curve in the model

Capital Demand and Supply



Calibration

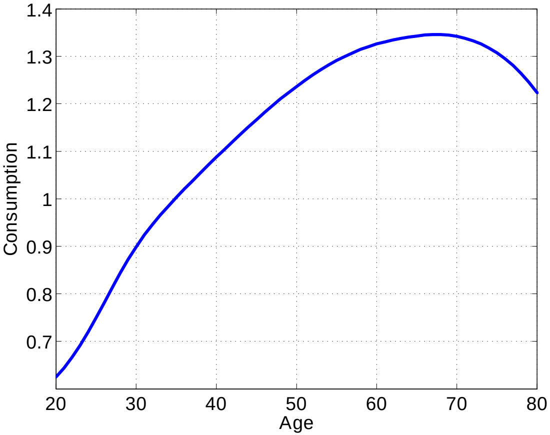
How to Calibrate Life Cycle Models

- Many empirical research: choose a target
  - $j^{\text{ret}} = 46$, $J = 81$
  - $\beta = 0.97$: Target is $K/Y \approx 3$
  - $\gamma = 2$: microeconomic evidences
  - Idiosyncratic income risks
    - Blundell et al. (2008,AER) etc.
    - $\rho = 0.98$, $\sigma_\kappa = 0.01$
  - Macroeconomic variables: Japanese economy
    - $\theta = 0.377$, $\delta = 0.08$
  - Age-efficiency profile: $\{\eta_j\}$
    - Hansen's (1993) method
  - Population distribution: $\{\mu_j\}$

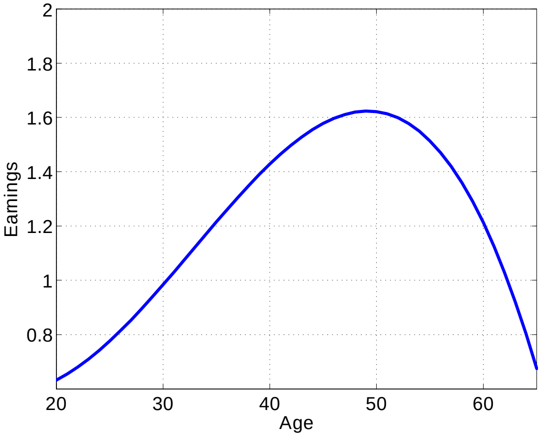
Numerical Results

- Numerical Results
  - Consumption and asset profiles: $\{c_j, a_j\}_{j=0}^{J}$
- Economic inequality: Storesletten et al. (2004,JME)
  - How much consumption insurance?
- Computation time (Fortran): 180 sec
  - #grid for policy function$= 100$
  - #grid for density function$= 10,000$
  - #grid for AR1 process$= 15$

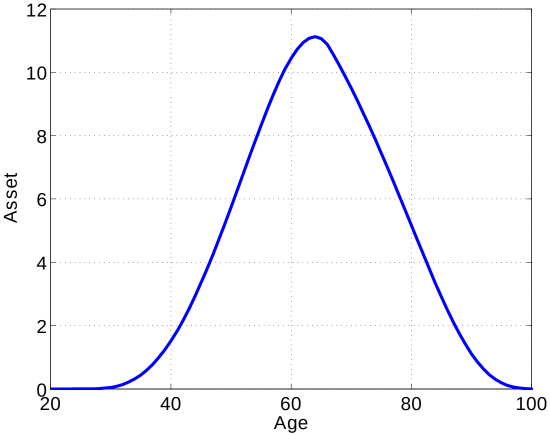
Consumption Profile



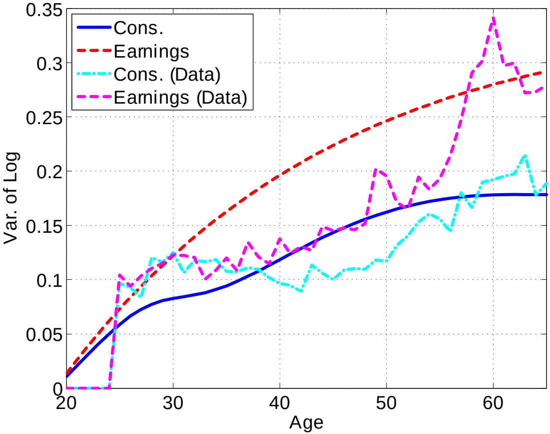
Earnings Profile



Asset Profile



Inequality over Life Cycle



Applications

- Endogenous labor supply:
  - Add intra-temporal FOC to the original problem

$$w\eta_j z_j = \frac{u'_h(c, 1-h)}{u'_c(c, 1-h)}$$

- Can we use the endogenous gridpoint method again?
  - Yes: Barillas and Fernandez-Villaverde (2007), Krueger and Ludwig (2006)

Applications (cont.)

- Economic inequality:
  - Huggett (1996,JME), Storesletten et al. (2004,JME), Heathcote et al. (2010,JPE)
  - Include transitory shocks, education background, marriage etc.
- Social security reforms:
  - Imrohoroglu et al. (1995,1997)
  - Need additional state variable for social security accounts
- Optimal taxation:
  - Conesa, Kitao and Krueger (2008)

$$\tau(y) = \tau_2 \left[y - \left(y^{-\tau_1} + \tau_0 \right)^{-\frac{1}{\tau_1}} \right]$$

And More...

- Two natural extension of life cycle models

1. Transition path
   - Social security reforms, tax reforms, aging etc.
2. Aggregate shock
   - Business cycle, asset pricing etc.

Transition

- Literature:
  - Conesa and Krueger (1999)
  - Nishiyama and Smetters (2007,2009)
- Why transition?
  - Welfare implications may differ between steady state comparison and considering transition path explicitly
- Why difficult?
  - Need to solve many generations problems

Transition (cont.)

- Basic ideas:
  - application of computing a steady state with *very long* backward induction
  - T (terminal period)$\rightarrow T-1, \cdots, \rightarrow 1$ (initial period)
  - Need computation time: several hours
- Transition path between two steady state
  - Without final steady state, it is impossible to compute the transition path (where to go?)
  - Initial steady state is not MUST (e.g., Auerbach and Kotlikoff, 1992)
  - But, you need to have an initial wealth distribution for each age, $\Phi_j(a, z)$: difficult to calculate

An Example

Yamada (2011,JEDC)

- Research question: How well does the life cycle model work?
  - Results: Macroeconomic variables and inequalities in Japan is, at least partially, explained by the standard life cycle model with macroeconomic and demographic changes
- Include deterministic TFP growth rate and demographic change in the model:

$$
\begin{aligned}
Y_t &= A_t K_t^{\theta} L_t^{1-\theta} \\
\mu_{j+1,t+1} &= \phi_{j,t}\mu_{j,t}
\end{aligned}
$$

Model (cont.)

Original Problem

- Bellman equation:

$$V_{j,t}(a,s) = \max_{c,a',h} \left\{ u(c_{j,t+j}, h_{j,t+j}) + \phi_{j,t} \beta \mathbb{E} V_{j+1,t+1}(a', s') \right\}$$

$$u(c_{j,t+j}, h_{j,t+j}) = \frac{[c_{j,t+j}^{\sigma} (\bar{h}_t - h_{j,t+j})^{1-\sigma}]^{1-\gamma}}{1-\gamma}$$

  - t: calender year, j: age
  - $h_{j,t+j}$: labor supply (endogenous), $\bar{h}_t$: time endowment

Model (cont.)

- Budget constraint:

$$c_{j,t} + a_{j+1,t+1} = (1 + (1 - \tau_t^{\text{cap}}) r_t)(a_{j,t} + b_t) + y_{j,t}$$

$$y_{j,t} = \begin{cases} (1 - \tau_t^{\text{ss}} - \tau_t^{\text{lab}}) w_t \eta_j e_j h_{j,t} \\ \varphi_t w_t L_t \end{cases}$$

  - $\{\tau_t^{\text{lab}}, \tau_t^{\text{cap}}, \tau_t^{\text{ss}}\}$: taxes for labor, capital and social security system

- Idiosyncratic labor productivity shock $s \equiv (\alpha, z, \varepsilon)$:

$$\begin{aligned} \ln e_t &= \alpha + z_j + \varepsilon_j, \\ \ln z_{j+1} &= \rho \ln z_j + \kappa_j \end{aligned}$$

Model (cont.)

- Aggregate economy:

$$K_t = \sum_{j=20}^{100} \mu_{j,t} \int a_{j,t} d\Phi_{j,t}(a,s)$$
$$L_t = \sum_{j=20}^{65} \mu_{j,t} \int \eta_j e_j h_{j,t+j} d\Phi_{j,t}(a,s)$$

Model (cont.)

- Social security system:

$$\sum_{j=20}^{65} \mu_{j,t} \int \tau_t^{ss} w_t \eta_j e_j h_{j,t+j} d\Phi_{j,t}(a,s) = \sum_{j=66}^{100} \mu_{j,t} \varphi_t w_t L_t.$$

- Government budget constraint:

$$G_t = \sum_{j=20}^{100} \mu_{j,t} \int \tau_t^{cap} r_t a_{j,t} d\Phi_{j,t}(a,s) + \sum_{j=20}^{65} \mu_{j,t} \int \tau_t^{lab} w_t \eta_j e_j h_{j,t+j} d\Phi_{j,t}(a,s)$$

Model (cont.)

Detrended Problem

- Detrend the problem:

$$v_{j,t}(\tilde{a}, s) = \max_{c,a',h} \left\{ u(\tilde{c}_{j,t+j}, \tilde{h}_{j,t+j}) + \phi_{j,t}\tilde{\beta}_t \mathbb{E} v_{j+1,t+1}(\tilde{a}', s') \right\}$$

$$u(\tilde{c}_{j,t+j}, \tilde{h}_{j,t+j}) = \frac{[\tilde{c}^{\sigma}_{j,t+j}(\bar{h}_t - \tilde{h}_{j,t+j})^{1-\sigma}]^{1-\gamma}}{1-\gamma}$$

  - $\tilde{c}_{j,t} = c_{j,t}/A_t^{1/(1-\theta)}$, $\tilde{a}_{j,t} = a_{j,t}/A_t^{1/(1-\theta)}$, $\tilde{h}_{j,t} = h_{j,t}$
  - $\tilde{\beta}_t = \beta(1+g_t)^{\sigma(1-\gamma)}$, $1 + g_t = A_{t+1}^{1/(1-\theta)}/A_t^{1/(1-\theta)}$

Model (cont.)

- Budget constraints of the detrended problem

$$\tilde{c}_{j,t} + (1+g_t)\tilde{a}_{j+1,t+1} = (1 + (1-\tau_t^{\text{cap}})r_t)(\tilde{a}_{j,t} + \tilde{b}_t) + \tilde{y}_{j,t}$$

$$\tilde{y}_{j,t} = \begin{cases} (1 - \tau_t^{\text{ss}} - \tau_t^{\text{lab}})\tilde{w}_t \eta_j e_j \tilde{h}_{j,t} \\ \tilde{\varphi}\tilde{w}_t \tilde{L}_t \end{cases}$$

Algorithm

1. Compute initial and final steady states: 1980 and 2200
2. Set an exogenous path of $\{g_t, \phi_t, \bar{h}_t\}_{t=1980}^{2200}$, and guess an equilibrium sequence of $\{r_t^0, \tilde{w}_t^0\}_{t=1980}^{2200}$.
3. Given the policy function of the final steady state, compute a sequence of policy functions using the EGM by backward induction
4. Given the policy functions, compute the distribution function from 1980 forwardly and compute aggregate variables, $\{r_t^1, \tilde{w}_t^1\}_{t=1980}^{2200}$
5. Check whether new factor prices $\{r_t^1, \tilde{w}_t^1\}$. are sufficiently close to the old ones $\{r_t^0, \tilde{w}_t^0\}$ for every year. If these are not close, update the price sequences and repeat steps $3-4$
6. If the factor prices are close in all periods, then stop!

Algorithm (cont.)

- In step 3, use the EGM again
- FOCs:

$$u_c'(\tilde{c}_{j,t}, \bar{h} - \tilde{h}_{j,t})$$
$$\geq \phi_{j,t}\tilde{\beta}_t \frac{(1 + (1 - \tau^{\text{cap}}_{t+1})r_{t+1})}{(1 + g_t)} \mathbb{E} u_c'(\tilde{c}_{j+1,t+1}, \bar{h} - \tilde{h}_{j+1,t+1})$$

$$\tilde{h}_{j,t} = \max\left[\bar{h}_t - \left(\frac{1-\sigma}{\sigma}\right)\frac{\tilde{c}_{j,t}}{(1 - \tau^{ss}_t)\tilde{w}_t \eta_j e_j}, 0\right]$$

Algorithm (cont.)

- Define RHS of the Euler equation:

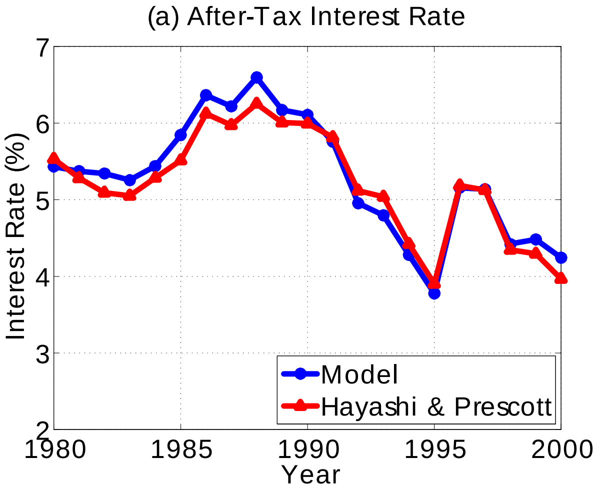
$$\Gamma'_{j,t}(\tilde{a}_j, s_j) = \frac{(1+(1-\tau^{\text{cap}}_{t+1})r_{t+1})}{(1+g_t)} \phi_{j,t}\tilde{\beta}_t E_j \left\{ \frac{[\tilde{c}^{\sigma}_{j+1,t+1}(\bar{h}_{t+1} - \tilde{h}_{j+1,t+1})^{1-\sigma}]^{1-\gamma}}{\tilde{c}_{j+1,t+1}} \right\}$$

- First order condition:

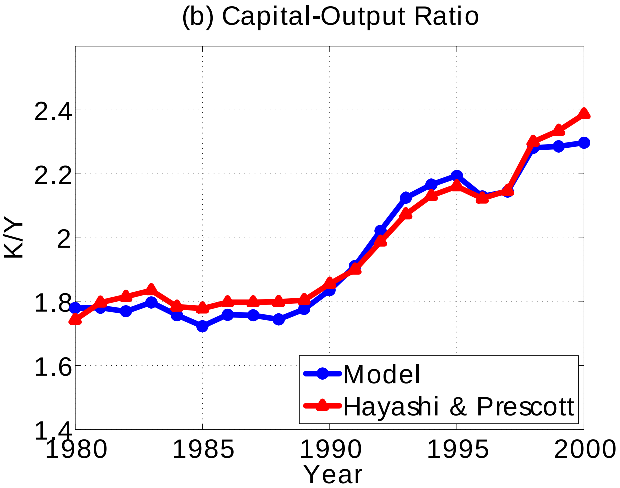
$$u'_c(\tilde{c}_{j,t}, \bar{h}_t - \tilde{h}_{j,t}) = \Gamma'_{j,t}(\tilde{a}', s)$$

  - Cobb-Douglas utility function is *invertible*

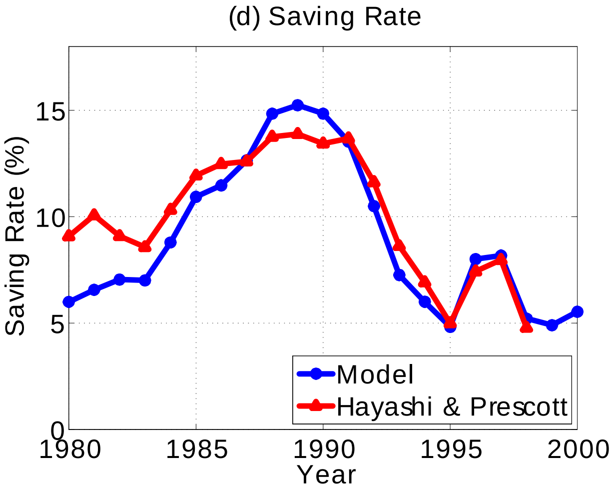
Example: Transition Path



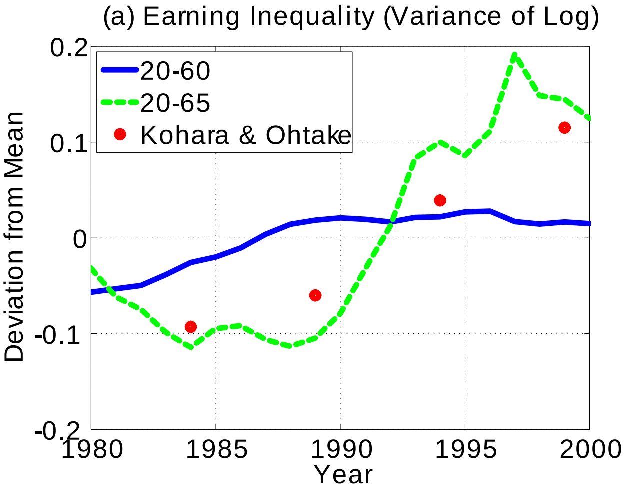
Example: Transition Path



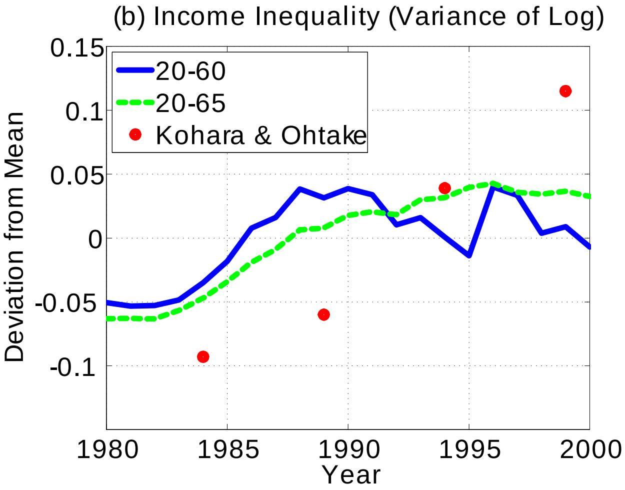
Example: Transition Path



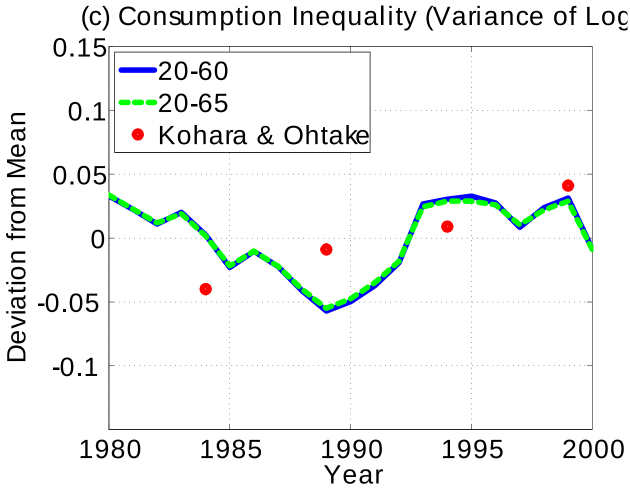
Example: Transition Path



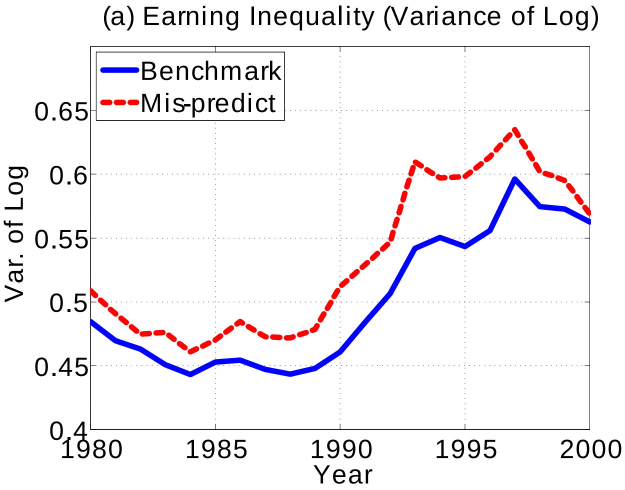
Example: Transition Path



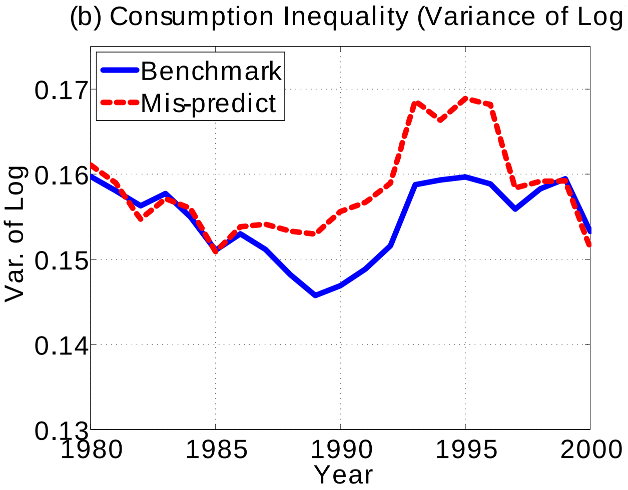
Example: Transition Path



Example: Transition Path



Example: Transition Path



Aggregate Shock

- Add aggregate shock (e.g., TFP shock) in the model
  - Storesletten et al. (2007,RED), Krusell et al. (2011)
- Why difficult?
  - Distribution affects factor prices:

$$u'\left(c_{j,t}\right) \geq \phi_j \beta \left(1 + r(K_{t+1})\right) \mathbb{E} u'\left(c_{j+1,t+1}\right)$$

  - Distribution function in the state space

$$V_j(a, z; \Phi) = \max \left\{ u(c_j) + \xi_j \beta \mathbb{E} V_{j+1}(a', z'; \Phi) \right\}$$

- Distribution function is *infinite*-dimensional objects $\Rightarrow$impossible to solve

Aggregate Shock

- Approximate aggregation
  - Krusell and Smith (1998), den Haan (1997)
  - Use not *exact* distribution function, but moments
- E.g., approximate prediction function:

$$\log K_{t+1} = \beta_0(A) + \beta_1(A) \log K_t$$

- Approximated Bellman equation:

$$V_j(a, z; K, A) = \max \left\{ u(c_j) + \xi_j \beta \mathbb{E} V_{j+1}(a', z'; K, A) \right\}$$

Another Approach to Solve OLG Models

- Add another state variables⇒Curse of dimensionality
  - E.g., physical asset and financial asset: $50 \times 50 = 2,500$
- Smolyak algorithm: sparse grid approximations
- Stochastic overlapping generations models
  - Krueger and Kubler (2003,2005)
  - Malin, Krueger and Kubler (2010)
  - Glover, Heathcote, Krueger and Ríos-Rull (2011)
- Points:
  - No idiosyncratic risks
  - Explicit portfolio choices

Advanced Topics

- Recent CPU have double/quad cores
  - Core2Duo, Core i7, Xeon etc.
- Use many cores simultaneously
  - MPI: need many PCs
  - Open MP: easy to implement
- Use Graphic Processors
  - Very fast, but need additional programming skills and tools

Reference List: General

- Judd, K. (1998): *Numerical Methods in Economics*, The MIT Press.
- Marimon, R. and A. Scott (1999): *Computational Methods for the Study of Dynamic Economies*, Oxford University Press.
- Miranda, M.J. and P.L. Fackler (2002): *Applied Computational Economics and Finance*, The MIT Press.
- Heer, B. and A. Maussner (2009): *Dynamic General Equilibrium Modeling, 2nd Edition*, Springer-Verlag.
- Press, W.H., S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery (2007): *Numerical Recipes: The Art of Scientific Computing, Third Edition*, Cambridge University Press.

Reference List: Projection Methods & VFI

- Judd, K. (1992): "Projection Methods for Solving Aggregate Growth Models," *Journal of Economic Theory*, 58, 410–452.
- McGrattan, E.R. (1996): "Solving the Stochastic Growth Model with a Finite Element Method," *Journal of Economic Dynamics and Control*, 20, 19–42.
- Christiano, L.J., and J.D.M. Fisher (2000): "Algorithms for Solving Dynamic Models with Occasionally Binding Constraints," *Journal of Economic Dynamics and Control*, 1179–1232.
- S.B. Aruoba, J. Fernández-Villaverde, and J.F. Rubio-Ramírez (2006): "Comparing Solution Methods for Dynamic Equilibrium Economies," *Journal of Economic Dynamics and Control*, 30, 2477–2508.

Reference List: EGM

- Carroll, C.D. (2006): "The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems," *Economics Letters*, 91, 312–320.
- Barillas, F., and J. Fernandez-Villaverde (2007): "A Generalization of the Endogenous Grid Method," *Journal of Economic Dynamics and Control*, 31, 2698–2712.
- Krueger, D. and A. Ludwig (2006): "On the Consequences of Demographic Change for Rates of Returns to Capital, and the Distribution of Wealth and Welfare," *Journal of Monetary Economics*, 54, 49–87.

Reference List: Life Cycle Models

- Huggett, M. (1996): “Wealth Distribution in Life-Cycle Economies," *Journal of Monetary Economics*, 38, 469–494.
- Storesletten, K., C. Telmer, and A. Yaron (2004): “Consumption and Risk Sharing over the Life Cycle," *Journal of Monetary Economics*, 51, 609–633.
- Conesa, J.C., S. Kitao, and D. Krueger (2008): “Taxing Capital? Not a Bad Idea After All!," *American Economic Review*, 99, 25–48.

Reference List: Transition

- Conesa, J.C. and D. Krueger (1998): "Social Security with Heterogeneous Agents," *Review of Economic Dynamics*, 2, 757-795.
- Nishiyama, S. and K. Smetters (2005): "Consumption Taxes and Economic Efficiency with Idiosyncratic Wage Shocks," *Journal of Political Economy*, 113, 1088–1115.
- Nishiyama, S. and K. Smetters (2007): "Does Social Security Privatization Produce Efficiency Gains?" *Quarterly Journal of Economics*, 122, 1677–1719.

Reference List: Aggregate Shock

- Krusel, P. and A.A. Smith Jr. (1998): "Incoe and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy*, 106(5), 867–896.
- Storesletten, K., C. Telmer, and A. Yaron (2007): "Asset Pricing with Idiosyncratic Risk and Overlapping Generations," *Review of Economic Dynamics*, 10(4), 519-548.
- den Haan (2010): "Assessing the Accuracy of the Aggregate Law of Motion in Models with Heterogeneous Agents," *Journal of Economic Dynamics and Control*, 34, 79–99.
- Krusell, P., T. Mukoyama, and A. Sahin (2011): "Asset Prices in a Huggett Economy," Journal of Economic Theory, 146, 812–844.

Reference List: AR(1)

- Tauchen, G. (1986): "Finite State Markov-Chain Approximations to Univariate and Vector Autoregresions," *Economics Letters*, 20, 177–181.
- Tauchen, G. and R. Hussey (1991): "Quadrature-Based Methods for Obtaining Approximate Solutions to Nonlinear Asset Pricing Models," *Econometrica*, 59, 371–396.
- Floden, M. (2008): "A Note on the Accuracy of Markov-Chain Approximations to Highly Persistent AR(1)-Processes," *Economics Letters*, 99, 516–520.

Reference List: Stochastic OLG

- Krueger, D. and F. Kubler (2003): "Computing Equilibrium in OLG Models with Stochastic Production," *Journal of Economic Dynamics and Control*, 28(7), 1411–1436.
- Malin, B., D. Krueger, and F. Kubler (2010): "Computing Stochastic Dynamic Economic Models with a Large Number of State Variables: A Description and Application of a Smolyak-Collocation Method," forthcoming, *Journal of Economic Dynamics and Control*.
- Glover, A., J. Heathcote, D. Krueger, and J.V. Ríos-Rull (2011): "Intergenerational Redistribution in the Great Recesion," *mimeo*, University of Pennsylvania.

Reference List: Accuracy

- den Haan, W. and A. Marcet (1994): “Accuracy in Simulations," *Review of Economic Studies*, 61, 3–17.
- Santos, M. (2000): “Accuracy of Numerical Solutions using the Euler Equation Residuals," *Econometrica*, 68(6), 1377–1402.
- Santos, M. and A. Peralta-Alva (2005): “Accuracy of Simulations for Stochastic Dynamic Models," *Econometrica*, 73(6), 1939–1976.
- Peralta-Alva, A. and M.S. Santos (2010): “Problems in the Numerical Simulation of Models with Heterogeneous Agents and Economic Distortions," *Journal of the European Economic Association*, 8(2-3), 617–625.

Reference List: Computers

- Aldrich, E.M., J. Fernández-Villaverde, A.R. Gallant, and J.F. Rubio-Ramírez (2010): "Tapping the Supercomputer Under Your Desk: Solving Dynamic Equilibrium Models with Graphics Processors," *mimeo*, University of Pennsylvania.
- Morosov, S. and S. Mathur (2010): "Massive Parallel Computation using Graphics Processors with Application to Optimal Experimentation in Dynamic Control," *mimeo*, Morgan Stanley.