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Can Cross-Border Financial Markets Create Endogenously Good Collateral in a Crisis?

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Abstract

In this paper, we explore whether markets can create endogenously good collateral in a crisis by analyzing a simple exchange economy where a country-specific catastrophic shock is shared between two countries. To see this possibility, we examine whether the equilibrium achieved by the time-0 complete markets with solvency constraints can be recovered in the dynamically complete markets with collateral constraints. This paper demonstrates that it is possible to recover the time-0 equilibrium outcome in a sequential manner when pricing errors occur randomly in evaluating Lucas trees at a catastrophic event. Such stochastic components may be interpreted as a policy initiative to create good collateral and yield constrained efficient outcomes at crisis periods.

Keywords: Solvency Constraints; Collateral Constraints; Dynamic Optimal Contract; Catastrophic Shocks
JEL classification: F34, G12, G15

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1 Introduction

In this paper, we explore whether cross-border financial markets can create endogenously good collateral when it is urgently needed in a crisis. In particular, we investigate which kind of policy interventions in financial markets at crisis periods would trigger the creation of good collateral, and yield efficient outcomes with bilateral borrowing and lending between countries.

Employing a simple two-country exchange economy setup, we first investigate how country-specific catastrophic shocks are shared between countries in the time-0 complete markets with solvency constraints. We then examine whether the time-0 equilibrium outcome can be recovered in a sequential setup where the transactions of Lucas trees and contingent claims are subject to collateral constraints. If it is impossible, then we analyze which kind of interventions in financial markets is required to recover the time-0 constrained efficient outcome in a sequential setup.

As discussed intensively in the literature, the setup of time-0 complete markets is extremely unrealistic, because it is hard to imagine that every financial contract is made *ex ante* when an economy starts in time 0. Thus, it is important to demonstrate that the time-0 equilibrium outcome can be achieved successfully in a sequential manner. Without any constraint or friction, it is quite possible to recover the time-0 complete markets outcome in dynamically complete markets. However, it may not be a case in the presence of some enforcement constraints. Nevertheless, most of the existing papers, including Kehoe and Perri (2002), Lustig and Nieuwerburgh (2005), Lustig (2007), and Chien and Lustig (2010), present only the time-0 complete markets outcome without any consideration of recovering the outcome in more realistic environment such as a sequential setup.[1]

This paper attempts to demonstrate the relevance of the time-0 constrained efficient outcome by specifying some conditions under which the time-0 equilibrium outcome may be recovered in a sequential manner. Following Lustig (2007), we construct a two-country exchange economy with solvency constraints. A *solvency constraint* requires

[1] Following Kehoe and Perri (1993), Alvarez and Jermann (2000) develop a sequential setup without having any long-lived asset. Kehoe and Perri (2004) decentralize constrained efficient allocations that arise from enforcement constraints.

that a net financial position should be non-negative at any state in any point of time; it is assumed that even current labor endowment of a debtor cannot be confiscated upon default. Here, we introduce a country-specific catastrophic shock such that solvency constraints may be severely binding.

One of the most substantial difference between a time-0 setup and a sequential setup is that a solvency constraint is much more stringent in the latter. In this paper, a solvency constraint in a sequential setup is called a *collateral constraint*. Given this stringent enforcement constraint, any one-period borrowing contracts and short positions in Lucas trees need to be backed by one-period contingent bonds or long positions in Lucas trees as collateral assets. Thus, it may be impossible for a damaged country to cover uninsured catastrophic losses by making only one-period financial contracts and Lucas trees without violating collateral constraints.

However, when pricing errors occur randomly in evaluating Lucas trees during a catastrophic event, it is possible to recover the time-0 equilibrium outcome in a sequential manner. More concretely, depending on the sign of pricing errors, either Lucas trees or contingent bonds are relatively risky in equilibrium. In addition, thanks to pricing errors, there emerge richness in risky bonds and cheapness in safe bonds. Then, a damaged country makes short positions in rich risky bonds and long positions in cheap safe bonds, thereby exploiting arbitrage profits. At the same time, a damaged country can satisfy collateral constraints, because long positions in safe bonds serves as collateral assets. Even a nondamaged country benefits from the above financial transactions with damaged country, because it can obtain an investment opportunity to smooth temporary relative gains over time.

In this way, the bilateral lending and borrowing in a sequential setup work to recover the time-0 constrained efficient outcome with random pricing errors associated with Lucas trees. We attempt to interpret realistically the above stochastic component as a sort of policy interventions, possibly initiated by a central bank. With such proper interpretations of the pricing error, we could consider the constrained efficient outcome delivered by the time-0 equilibrium as a reasonable and realistic equilibrium even when solvency constraints are severely binding.

This paper is organized as follows. Section 2 presents both a time-0 setup with solvency constraints and a sequential setup with collateral constraints, while Section 3 presents the calibration results. Section 4 offers concluding comments.

2 Model

Following the framework proposed by Kehoe and Perri (2002), Lustig (2007), and Chien and Lustig (2010), we construct a two-country exchange economy with solvency constraints first in a time-0 setup, and then put it in the context of a sequential setup with collateral constraints.

The labor endowment of each country is subject to country-specific catastrophic shocks on the level of the labor endowment. On the other hand, dividends on Lucas trees are proportional to the world endowment. In terms of market structures, markets are complete with respect to country-specific catastrophic shocks.

However, each country is subject to solvency constraints in the sense that net financial positions cannot be negative in every possible future state. This constraint is motivated by the fact that it is difficult for even current labor endowment of a debtor to be confiscated upon default.

2.1 Time-0 complete markets with solvency constraints

2.1.1 Labor endowment and Lucas trees

A world economy consists of infinite-horizon exchange economies of country i ($i = 1$ or 2) in a discrete time setup. It is assumed that the labor endowment is homogeneous within each country, but heterogeneous between the two countries. Each country receives labor endowments subject to country-specific catastrophic shocks. There is a fixed supply of Lucas trees whose dividends are subject to world common shocks. The quantity of Lucas trees is standardized to one.

A set of states of country-specific labor endowment is defined as $y \in Y = \{y_1, \ldots, y_m\}$, while a set of states of dividends on Lucas trees is denoted as $z \in Z = \{z_1, \ldots, z_n\}$. A combination of country-specific and world common states is expressed by $s_t = (y_t, z_t)$, where s_t is in $S = Y \times Z$. In addition, $s^t = (y^t, z^t)$ denotes a history from time 0 up to time t, while $s^j \succeq s^t$ represents a continuation history from s^t.

Furthermore, we assume that the dividend on Lucas trees $d(z)$ is proportional to the total labor endowment $(e^1(y) + e^2(y))$. Therefore, a combination of country-specific shocks constitutes aggregate states. Hereafter, $e^i(y_t)$ denotes the labor endowment of country i, and $d(z_t)$ denotes the dividend on Lucas trees. The transition probability of the above state variables $\pi(y', z'|y, z)$ evolves according to the following Markov process: $\pi(z'|z) = \sum_{y' \in Y} \pi(y', z'|y, z)$, $\forall z \in Z$, $\forall y \in Y$. Given the above processes of labor endowment and dividends, optimal policy functions of consumption and portfolios depend on state z_t.

2.1.2 Preferences and resource constraints

Country i maximizes expected lifetime utility with respect to consumption at state s^t $(c^i(s^t))$ as follows:

$$U(\{c^i(s^t)\}_{t=0}^{\infty})(s_0) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t|s_0) u\left[c^i(s^t)\right], \ i \in \{1, 2\},$$

where a preference is characterized as utility with constant relative risk aversion, or $u\left[c^i(s^t)\right] = \frac{c^i(s^t)^{1-\gamma}}{1-\gamma}$, γ denotes the degree of relative risk aversion, and β represents the rate of time preference.

The resource constraint of the world economy is given by:

$$e(z_t) = e^1(y_t) + e^2(y_t) + d(z_t).$$

Hereafter, α denotes the ratio of dividends to total labor endowment:

$$\alpha = \frac{d(z_t)}{e(z_t)}.$$

2.1.3 Time-0 complete markets with solvency constraints

In this economy with time-0 complete markets, both the shares of Lucas trees and one period contingent claims are traded between the two countries. $\theta^i(s^t)$ denotes the share of Lucas trees held by country i in time t, while $a^i(s^0, s^{t+1})$ represents the time-0 holding of claims on one unit of goods at state $s^t \in S^t$, and $p(z^t)$ is the price of Lucas trees,.

The market clearing conditions hold as follows:

$$\theta^1(s^t) + \theta^2(s^t) = 1, \tag{1}$$

$$a^1(s^0, s^t) + a^2(s^0, s^t) = 0, \text{ for all } s^t \in S^t. \tag{2}$$

As mentioned above, even current labor endowment of a debtor cannot be confiscated upon default. Therefore, the net position of financial assets cannot be negative at any state in any point of time:

$$\left[p(z^t) + d(z_t)\right] \theta^i(s^{t-1}) \geq -a^i(s^0, s^t), \ \forall s^t \in S^t. \tag{3}$$

We call the above enforcement constraint a *solvency constraint.*

2.2 Construction of a representative agent model with time-varying Negishi weights

2.2.1 Time-0 cost minimization problem

Following Lustig (2007), we thus construct a representative agent model with time-varying Negishi weights (Negishi, 1960) in time-0 setup, and compute the constrained competitive equilibrium using stochastic discount factors derivable from the representative agent model. For this end, we convert the time-0 utility maximization problem to its dual problem or the time-0 cost minimization problem together with a single promise-keeping constraint, and the solvency constraints, both of which are defined below.

The construction of the time-0 cost minimization problem greatly simplifies the computation procedure of the constrained equilibrium for the following reasons. First, the value function represented by a promise-keeping constraint can summarize a history of the realized states, and serve as a state variable; consequently, the space of state variables is reduced substantially. Second, Negishi weights can be computed from the cumulation of the Lagrange multipliers associated with solvency constraints. Because the Lagrange multiplier is positive at a default state and zero otherwise, Negishi weights become time-varying depending on whether solvency constraints are binding.

A life-time budget constraint of country i is rewritten as:

$$\sum_{t\geq 0}\sum_{s^t\in S^t} Q(s_0,s^t)\left[c^i(s^t)-e^i(y_t)\right]\leq w_0^i, \tag{4}$$

where w_0^i is the initial endowment, $q\left(s^{t-1},s^t\right)$ corresponds to a stochastic discount factor between state s^{t-1} and state s^t, and $Q(s_0,s^t)=q\left(s_0,s^1\right)\cdot q\left(s^1,s^2\right)\cdots q\left(s^{t-1},s^t\right)$.

Given equation (4), we reformulate a solvency constraint (3) as follows. If equation (3) is binding and the net financial asset is zero upon default at state s^t, then consumption from state s^t on has to be financed by only the current and future labor endowment. Accordingly, lifetime consumption is equal to lifetime labor endowment at state s^t:

$$\sum_{j\geq t}\sum_{s^j\succeq s^t} Q(s^t,s^j)c^i\left(s^j\right)=\sum_{j\geq t}\sum_{s^j\succeq s^t} Q(s^t,s^j)e^i\left(y_j\right).$$

Conversely, if collateral constraints are not binding and the net financial asset is still positive, then:

$$\sum_{j\geq t}\sum_{s^j\succeq s^t} Q(s^t,s^j)c^i\left(s^j\right)>\sum_{j\geq t}\sum_{s^j\succeq s^t} Q(s^t,s^j)e^i\left(y_j\right).$$

Employing the above life-time budget constraint and solvency constraints, we can characterize the time-0 problem as follows:

$$\max_{\{c^i\}}\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi(s^t|s_0)u\left[c^i(s^t)\right],$$
$$\text{s.t.}\ \sum_{t\geq 0}\sum_{s^t\in S^t} Q(s_0,s^t)\left[c^i(s^t)-e^i(y_t)\right]\leq w_0^i,$$
$$\sum_{j\geq t}\sum_{s^j\succeq s^t} Q(s^t,s^j)c^i\left(s^j\right)\geq\sum_{j\geq t}\sum_{s^j\succeq s^t} Q(s^t,s^j)e^i\left(y_j\right),\ \forall s^t\in S^t,\ t\geq 0.$$

If a country is in default at a certain state, then the last constraint (solvency constraint) is binding.

The dual problem to the above time-0 problem, that is, the cost minimization problem to attain lifetime utility v_0^i in time 0, is characterized as follows:

$$C^i\left(s_0\right) = \inf_{\{c^i\}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} Q(s_0, s^t) c^i(s^t), \tag{5}$$

$$\text{s.t.} \sum_{t \geq 0} \sum_{s^t \in S^t} \beta^t \pi(s^t | s_0) u\left[c^i(s^t)\right] = v_0^i, \tag{6}$$

$$\sum_{j \geq t} \sum_{s^j \succeq s^t} Q(s^t, s^j) c^i\left(s^j\right) \geq \sum_{j \geq t} \sum_{s^j \succeq s^t} Q(s^t, s^j) e^i\left(y_j\right), \ \forall s^t \in S^t, \ t \geq 0. \tag{7}$$

The second equation is called a *promise-keeping constraint* in the sense that the optimal solution allows a consumer to attain at least lifetime utility v_0^i. As mentioned before, v_0^i can summarize the history of realized states, and economize the space of state variables.

2.2.2 Time-varying Negishi weights

In the above cost minimization problem, the Lagrange multiplier μ_0^i is assigned to the promise-keeping condition (6), while the multipliers $\tau^i(s^t)$ are associated with the solvency constraints (7) state by state. The multiplier $\tau^i(s^t)$ may be either zero or positive depending on whether a solvency constraint is binding. Using these multipliers, we rewrite the cost minimization problem (5) as:

$$C^i\left(s_0\right) = \inf_{\{c^i\}} \left\{ \begin{array}{c} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} Q(s_0, s^t) c^i(s^t) \\ +\mu_0^i \left[v_0^i - \sum_{t \geq 0} \sum_{s^t \in S^t} \beta^t \pi(s^t|s_0) u\left[c^i(s^t)\right]\right] \\ +\sum_{t=0}^{\infty} \sum_{s^t \in S^t} Q(s_0, s^t) \left[\tau^i(s^t)\left[\sum_{j \geq t} \sum_{s^j \succeq s^t} Q(s^t, s^j) e^i\left(y_j\right) - \sum_{j \geq t} \sum_{s^j \succeq s^t} Q(s^t, s^j) c^i\left(s^j\right)\right]\right] \end{array} \right\}.$$

Exploiting a technique presented by Marcet and Marimon (1999),[2] we define the cumulative multiplier $\chi^i(s^t)$ as $\chi^i(s^t) \equiv \chi^i(s^{t-1}) - \tau^i(s^t)$ given $\chi_{-1}^i = 1$,[3] and further rewrite the above cost minimization problem as:

$$C^i\left(s_0\right) = \inf_{\{c^i\}} \left\{ \begin{array}{c} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} Q(s_0, s^t) \left[\chi^i(s^t) c^i(s^t) - \tau^i(s^t) \sum_{j \geq t} \sum_{s^j \succeq s^t} Q(s^t, s^j) e^i\left(y_j\right)\right] \\ +\mu_0^i \left[v_0^i - \sum_{t \geq 0} \sum_{s^t \in S^t} \beta^t \pi(s^t|s_0) u\left[c^i(s^t)\right]\right] \end{array} \right\}.$$

[2]Marcet and Marimon (1992) use the same technique. Messner and Pavoni (2004) presented some cases in which Marcet and Marimon (1999) may not work.

[3]If Lucas trees are equally endowed in time 0, then $\chi_0^1 = \chi_0^2 = 1$.

The first order condition with respect to $c^i(s^t)$ in the above cost minimization problem is

$$\frac{\mu_0^i}{\chi^i(s^t)} u'\left[c^i(s^t)\right] = \frac{Q(s_0, s^t)}{\beta^t \pi(s^t|s_0)}.$$

Because the right hand side of the above equation is independent of i, we have

$$\frac{\mu_0^1}{\chi^1(s^t)} u'\left[c^1(s^t)\right] = \frac{\mu_0^2}{\chi^2(s^t)} u'\left[c^2(s^t)\right],$$

or

$$\zeta^1(s^t) u'\left[c^1(s^t)\right] = \zeta^2(s^t) u'\left[c^2(s^t)\right], \tag{8}$$

where $\zeta^i(s^t)$ is defined as $\frac{\mu_0^i}{\chi^i(s^t)}$.

In a representative agent framework (a planner's problem), Negishi weights are assigned to each lifetime utility, and correspond to the ratio of period marginal utility between the two agents (countries in our context). Thus, as equation (8) implies, $\zeta^1(s^t)$ and $\zeta^2(s^t)$ can be used as Negishi weights. As Lustig (2007) demonstrates, the time-0 planner's objective (a representative agent model) is formulated as:

$$\max_{\{c^1, c^2\}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t|s_0) \left[\zeta^1(s^t) u\left[c^1(s^t)\right] + \zeta^2(s^t) u\left[c^2(s^t)\right]\right].$$

Since a period preference is $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $\left[\frac{c^2(s^t)}{c^1(s^t)}\right]^{\gamma} = \frac{\zeta^2(s^t)}{\zeta^1(s^t)}$ holds. Given that $h(s^t) \equiv \zeta^1(s^t)^{\frac{1}{\gamma}} + \zeta^2(s^t)^{\frac{1}{\gamma}}$, the consumption of country i is derived as $c^i(s^t) = \frac{\zeta^i(s^t)^{\frac{1}{\gamma}}}{h(s^t)} e(z_t)$. In addition, $\omega^i\left(s^t\right) \equiv \frac{\zeta^i(s^t)^{\frac{1}{\gamma}}}{h(s^t)}$ corresponds to the consumption share of each country.

Without any solvency constraint, Negishi weights are constant over time. Accordingly, the cross-country consumption share does not change over time at all. With solvency constraints, however, the consumption share between the two countries may fluctuate. The Negishi weight $\zeta^i(s^t)$ is constant unless country i is in default at state s^t, but otherwise, it is revised upward as a result of positive $\tau^i(s^t)$ (the multiplier associated with a solvency constraint). Therefore, the consumption share of country i increases when country i is in default at state s^t. One country subject to a solvency constraint at state s^t cannot transfer resources from state s^t to any state which is realized earlier. Consequently, the consumption share of the corresponding country at state s^t becomes large relative to the share of a previous state where solvency constraint is not binding. The country in constraint yields higher consumption growth toward a state in which a solvency constraint is binding.

Here, $g(s^{t+1})$ denotes the growth of $h(s^t) = \zeta^1(s^t)^{\frac{1}{\gamma}} + \zeta^2(s^t)^{\frac{1}{\gamma}}$ (the sum of nonlinearly transformed Negishi weights) from state s^t to state s^{t+1}, or

$$g(s^{t+1}) \equiv \frac{h(s^{t+1})}{h(s^t)}.$$

By construction, $g(s^{t+1})$ is one or higher. A higher $g(s^{t+1})$ implies that either of the two countries face severer solvency constraints between time s^t and time s^{t+1}. Lustig (2007) called $g(s^{t+1})$ *liquidity shocks*.

As demonstrated by Lustig (2007),[4] thanks to a complete markets setup, a stochastic discount factor between state s and state s' can be defined as a function of the aggregate endowment and the above liquidity shock, or

$$\pi(s'|s) \left(\frac{e(z')}{e(z)}\right)^{-\gamma} g(s'|s)^{\gamma}. \tag{9}$$

Without any solvency constraint ($g(s'|s) = 1$), a stochastic discount factor reduces to a standard one or $\pi(s'|s) \left(\frac{e(z')}{e(z)}\right)^{-\gamma}$.

2.2.3 Asset pricing in solvency-constrained economy

Once a solvency constraint is binding at a certain state in time $t+1$ (s^{t+1}), then

$$q(s^t, s^{t+1}) u' \left[c^i(s^t)\right] - \beta\pi(s_{t+1}|s_t) u' \left[c^i(s^{t+1})\right] > 0 \tag{10}$$

holds as intertemporal efficiency conditions. These inequalities are often called *Euler inequalities*.

The Euler inequality implies that the stochastic discount factor ($\beta\pi(s_{t+1}|s_t)\frac{u'\left[c^i(s^{t+1})\right]}{u'\left[c^i(s^t)\right]}$, hereafter, SDF) of a country subject to a collateral constraint at a certain state in time

[4] In Lustig (2007), when the aggregate endowment declines (that is, $\left(\frac{e(z')}{e(z)}\right)^{-\gamma}$ is larger), more consumers face solvency constraints as a result of more volatile idiosyncratic shocks (that is, $g(s'|s)^{\gamma}$ is larger). Accordingly, stochastic discount factors tend to correlated heavily negatively with dividends on Lucas trees in a future recession state; this is a source of a larger risk premium in his model.

$t+1$ becomes irrelevant to the asset pricing behavior in time t. From equation (10), we have

$$\beta\pi(s_{t+1}|s_t)\frac{u'\left[c^i(s^{t+1})\right]}{u'\left[c^i(s^t)\right]} < \beta\pi(s_{t+1}|s_t)\frac{u'\left[c^{i'}(s^{t+1})\right]}{u'\left[c^{i'}(s^t)\right]} = q(s^t, s^{t+1}) \tag{11}$$

for a constrained country (country i) and a unconstrained country (country i'). Thus, the SDF of a unconstrained country is larger than that of a constrained country in equilibrium. Note that either country satisfies Euler equation at any state in any point of time.

2.3 Numerical procedures

The construction of a representative agent model with time-varying Negishi weights helps to substantially simplify the numerical computation procedure. In particular, once the revision rule of Negishi weights is established, it is possible to compute a stochastic discount factor between state s^t and state s_0 ($Q(s_0, s^t)$) by equation (9). Then, we can pin down the equilibrium path of the consumption share of each country and asset pricing without solving any individual optimization problem including optimal portfolio problems.

Thus, the derivation of the revision rule of Negishi weights plays an essential role in the numerical procedure. While the appendix reviews the numerical method in detail, a key idea is conceptually simple. To begin with, we compute the consumption share that satisfies a solvency constraint or equation (7) for every one-period ahead state s' for country i, denoted by $\underline{\omega}^i(s')$. If the current consumption share $\omega^i(s)$ is smaller than $\underline{\omega}^i(s')$, then a solvency constraint is regarded as binding at state s', and the Negishi weight for a constrained country is revised upward from state s onto state s'. More concretely, if $\omega^i(s) < \underline{\omega}^i(s')$ at state s' for country i, then the Negishi weight of country i is revised upward as $\zeta^i(s') = \left[\underline{\omega}^i(s')h(s')\right]^\gamma$, where $h(s') \equiv \zeta^1(s')^{\frac{1}{\gamma}} + \zeta^2(s')^{\frac{1}{\gamma}}$.[5]

[5] In the numerical procedure described in the appendix, we use as Negishi weights $\omega^i(s)$ instead of $\zeta^i(s)$ after all variables are standardized by the total world endowment $e(z_t)$.

2.4 Sequential trading with collateral constraints

2.4.1 Collateral constraints

This subsection presents a sequential setup. The most essential difference between a time-0 setup and a sequential setup is that a solvency constraint or equation (3) is much more stringent in the latter. That is, equation (3) is rewritten as follows:

$$\left[p(z^{t+1}) + d(z_{t+1})\right] \theta^i(s^t) \geq -a^i(s^t, s^{t+1}), \ \forall s^{t+1} \in S^{t+1}. \tag{12}$$

We call the above enforcement constraint a *collateral constraint.* This formulation of solvency constraints implies that the net position of financial portfolio consisting of only one-period contingent bonds and Lucas trees must be nonnegative in every possible one-period ahead state. When equation (12) is binding, a debtor country is indifferent between default with confiscation and full repayment at maturity. In other words, the outstanding liability in short positions is enforceable up to the value of financial assets as collateral. As discussed later, a collateral constraint severely limits the borrowing ability of each country.

If a collateral constraint or equation (12) is binding for country i, then the possessed financial assets are exhausted for repayment (or they are confiscated), and the next period's wealth ($w^i(s^{t+1})$) consists of only the labor endowment:

$$w^i(s^{t+1}) = e^i(y_{t+1}),$$

otherwise, it is equal to:

$$w^i(s^{t+1}) = e^i(y_{t+1}) + \left[p(z^{t+1}) + d(z_{t+1})\right] \theta^i(s^t) + a^i(s^t, s^{t+1}).$$

Then, a sequential budget constraint is written as:

$$c^i(s^t) + p(z^t)\theta^i(s^t) + \sum_{s^{t+1\prime} \in S^{t+1}} q(s^t, s^{t+1}) a^i(s^t, s^{t+1}) \leq w^i(s^t). \tag{13}$$

Given the above collateral constraint (12), each country maximizes expected lifetime utility subject to budget constraint (13):

$$\max_{\{c^i\},\{\theta^i\},\{a^i\}} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t|s_0) u \left[c^i(s^t) \right], \tag{14}$$

$$\text{s.t. } c^i(s^t) + p(z^t)\theta^i(s^t) + \sum_{s^{t+1} \in S^{t+1}} q(s^t, s^{t+1}) a^i(s^t, s^{t+1}) \leq w^i(s^t),$$

$$\left[p(z^{t+1}) + d(z_{t+1}) \right] \theta^i(s^t) \geq -a^i(s^t, s^{t+1}), \ \forall s^{t+1} \in S^{t+1}.$$

A collateral constrained competitive equilibrium is defined as follows. Given the initial wealth $\{w_0^1, w_0^2\}$, the trading strategy $\{a^i(s^t, s^{t+1})\}$, $\{c^i(s^t)\}$, $\{\theta^i(s^t)\}$, the pricing function $\{q(s^t, s^{t+1})\}$ and $\{p(z^t)\}$, each country maximizes (14) subject to equations (13) and (12), and market clearing conditions (1) and (2) are satisfied.

We make some remarks on the above type of collateral constraints. First, an insurer country has to back catastrophe insurance payments $(-a^i(s^t, s^{t+1}))$ by his holdings of Lucas trees $(\left[p(z^{t+1}) + d(z_{t+1}) \right] \theta^i(s^t))$. In other words, a country can offer catastrophe insurance capacity only up to the value of Lucas trees at hand. Second, equation (12) does not impose any upper limit on short positions in contingent claims. A country can issue contingent bonds as long as he can repay bond obligations by Lucas trees as collateral. Third, equation (12) does not exclude any short position in Lucas trees. A country can make short positions as long as he carries long positions in contingent bonds. Here, we assume that any short position is settled by cash or netting; that is, any delivery of Lucas trees is not involved in trading short positions. In this regard, making short positions in Lucas trees may be interpreted as issuing contingent bonds whose repayment is proportional to the price of Lucas trees.

2.4.2 Borrowing restrictions imposed by collateral constraints

We finally point out that a collateral constraint or equation (12) is much more stringent than a solvency constraint or equation (3) in the sense that the former extremely constrains the borrowing ability. It is easy to show that one country cannot borrow from the other country in net without violating collateral constraints. The net amount financed from country 1 by country 2 in state s^t is equal to:

$$-p(z^t)\theta^2(s^t) - \sum_{s^{t+1}\succeq s^t} q(s^t, s^{t+1})a^2(s^t, s^{t+1}). \tag{15}$$

Substituting the arbitrage pricing of Lucas trees or $p(z^t) = \sum_{s^{t+1}\succeq s^t} q(s^t, s^{t+1}) \left[p(z^{t+1}) + d(z_t)\right]$ to the above equation, we obtain:

$$\begin{aligned} & -p(z^t)\theta^2(s^t) - \sum_{s^{t+1}\succeq s^t} q(s^t, s^{t+1})a^2(s^t, s^{t+1}) \qquad (16) \\ & = -\sum_{s^{t+1}\succeq s^t} q(s^t, s^{t+1}) \left[a^2(s^t, s^{t+1}) + \left[p(z^{t+1}) + d(z_{t+1})\right]\theta^2(s^t)\right]. \end{aligned}$$

As long as a collateral constraint (12) holds, the right hand side of the above equation cannot be positive at all. In other words, if there emerge net flows of funds from one country to the other in a time-0 setup with binding solvency constraints, then it is impossible to restore the time-0 equilibrium outcome in a sequential manner.

2.4.3 Random errors in pricing Lucas trees

As demonstrated above, given a collateral constraint or equation (12), one country cannot borrow a positive amount of resources from the other country. Accordingly, if there emerges a net flow of funds from country 1 (a creditor country) to country 2 (a debtor country) at state s^t in the time-0 constrained equilibrium, then it is impossible to recover the time-0 equilibrium outcome in a sequential manner.

We here introduce pricing errors associated with Lucas trees in order to relax the extent that collateral constraints are binding in a sequential setup. More concretely, the price of Lucas trees ($p(z^t)$) deviates from the arbitrage pricing by a random variable ϵ_t as follows:

$$\hat{p}(z^t) = \sum_{s^{t+1}\succeq s^t} q(s^t, s^{t+1}) \left[p(z^{t+1}) + d(z_{t+1})\right] + \epsilon_t. \tag{17}$$

When ϵ_t satisfies $E_{t-1}\epsilon_t = 0$, the arbitrage condition still holds prior to time t: that is, we still have $E_{t-1}\left[p(z^t)\right] = E_{t-1}\left[\sum_{s^{t+1}\succeq s^t} q(s^t, s^{t+1}) \left[p(z^{t+1}) + d(z_{t+1})\right]\right]$. In the concluding section, we interpret this pricing error associated with Lucas trees as policy interventions in financial markets.

Substituting equation (17) into equation (15) where country 2 is a debtor country, we obtain:

$$
\begin{aligned}
&- p(z^t)\theta^2(s^t) - \sum_{s^{t+1}\succeq s^t} q(s^t, s^{t+1})a^2(s^t, s^{t+1}) \\
&= -\epsilon_t\theta^2(s^t) - \sum_{st+1\succeq s^t} q(s^t, s^{t+1}) \left[a^2(s^t, s^{t+1}) + \left[p(z^{t+1}) + d(z_{t+1})\right]\theta^2(s^t)\right]. \qquad (18)
\end{aligned}
$$

As equation (18) implies, when Lucas trees are over-evaluated ($\epsilon_t > 0$), having short positions in Lucas trees ($\theta^2(s^t) < 0$) may generate net positive funding with a collateral constraints satisfied ($-\epsilon_t\theta^2(s^t) > 0$). In this case, long positions need to be constructed for a portfolio of one-period contingent claims; otherwise, a collateral constraint cannot be satisfied.

Conversely, when Lucas trees are under-evaluated ($\epsilon_t < 0$), having long positions in Lucas trees ($\theta^2(s^t) > 0$) and short positions in contingent claims may generate net positive funding with a collateral constraints satisfied ($-\epsilon_t\theta^2(s^t) > 0$). In either case, given a deviation from arbitrage pricing, having short positions in rich assets, which are Lucas trees if $\epsilon_t > 0$, and long positions in cheap assets, which are Lucas trees if $\epsilon_t < 0$ may lead to net positive funding even if a collateral constraint is binding.

At the same time, country 1 (a creditor country) benefits from the over- or under-evaluation, because country 1 can smooth consumption over time by exploiting the investment opportunities that are offered by country 2. That is, the random pricing errors of Lucas trees would be beneficial for both countries (a debtor country and a creditor country).

2.5 Derivation of portfolio positions in a sequential setup

In standard representative agent models, optimal portfolio problems are implicit in solving equilibrium paths, and are often considered as trivial issues. A major reason for this is that a portfolio problem is reduced to a simple allocation of market portfolios and non-contingent bonds when any constraint other than resource constraints is absent. As mentioned in the previous subsection, portfolio problems are also implicit in solving our planner's problem. However, they are potentially important when solvency constraints are present, because there may emerge complicated financial transactions between a country damaged by catastrophic shocks and a nondamaged country.

Thanks to a two-country setup, it is possible to recover the portfolio positions of country 1 and country 2 as follows. From the budget constraint (13), we obtain the following system of equations to determine portfolio rules together with the market clearing conditions (1) and (2):

$$c^1(s^t)+p(z^t)\theta^1(s^t)+\sum_{s^{t+1}\in S^{t+1}} q(s^t,s^{t+1})a^1(s^t,s^{t+1}) = e^1(y_t)+\left[p(z^t)+d(z_t)\right]\theta^1(s^{t-1})+a^1(s^{t-1},s^t),$$

and

$$c^2(s^t)+p(z^t)\theta^2(s^t)+\sum_{s^{t+1}\in S^{t+1}} q(s^t,s^{t+1})a^2(s^t,s^{t+1}) = e^2(y_t)+\left[p(z^t)+d(z_t)\right]\theta^2(s^{t-1})+a^2(s^{t-1},s^t).$$

Note that both consumption and asset prices are standardized by the total endowment.

It is possible to identify from simulation results which state and which country faces a solvency constraint. These identified facts simplify the above system of equations. When a solvency constraint is binding on country 1 in state s^t in time t, country 1 repays up to:

$$-a^1(s^{t-1},s^t) = \left[p(z^t)+d(z_t)\right]\theta^1(s^{t-1}).$$

When a solvency constraint is binding on country 2 in state s^t in time t, country 1 is repaid by:

$$a^1(s^{t-1},s^t) = \left[p(z^t)+d(z_t)\right](1-\theta^1(s^{t-1})).$$

After simplifying the system, we approximate portfolio rules by $\theta^i(s^t)=\nu_0^i+\nu^i c^i(s^t)$ and $a^i(s^t,s^{t+1}) = \alpha_0^i+\alpha^i c^i(s^t)$. Given the simulated series of asset prices and consumption shares, we identify the values of ν_0^i, ν^i, α_0^i, and α^i that minimize the sum of squared residuals of the above system for a certain range of $c^i(s^t)$. In so doing, we classify current states (time t states) into three possible states, including (1) neither country 1 nor country 2 receives adverse shocks, (2) only country 1 receives shocks, and (3) only country 2 receives shocks.

We may have a special and convenient case in which as of time $t-1$, either country would be subject to solvency constraints in any possible state of a one-period ahead period (time t). In this case, binding solvency constraints can identify portfolio positions precisely, and we can obtain exact positions without using any approximation. Indeed, the calibration results presented in Section 3 do not require using any approximation.

3 Calibration Exercises

3.1 Setup

This section explores numerically how the time-0 equilibrium outcome can be recovered in a sequential setup when pricing errors randomly occurs in evaluating Lucas trees. We first determine the size of country-specific catastrophic shocks following the existing empirical literature. Using US data for the period between 1869 and 1985, Cecchetti, Lam, and Mark (1990) identify catastrophic shocks on GDP. In their estimation, total annual output declines by 15.1% in the catastrophic regime, while it grows by 2.5% in the normal regime. The normal regime moves to the catastrophic regime with probability of 1.8% per year. Once the economy enters the catastrophic state, the state repeats itself with probability 51.0%.

On the other hand, Barro (2006) argues that the annual probability of catastrophic states is around 1.7%, and that the loss amounts to 15% through 64% of total output through intensively collecting data of developed and developing countries. These papers find that such catastrophic shocks permanently reduce the level of national output.

While catastrophic shocks may be persistent or even permanent as documented empirically,[6] we focus on a case with purely transitory country-specific catastrophic shocks (i.i.d. shocks) for a computational reason. Following the above findings, we assume that the labor endowment of a country declines by 20% with probability 1.8% per year. Without the realization of catastrophic shocks, the labor endowment remains at a given level. A catastrophic shock is assumed to be country-specific and uncorrelated between the two countries.

We treat cases where solvency constraints are severely binding by making the ratio of dividends to the world labor endowment (α) rather low. In time 0, both labor endowment

[6] As discussed in Gourio (2008), catastrophically damaged countries often experienced eventual recoveries.

and Lucas trees are equally distributed between the two countries. The rate of time preference is 5% ($\beta = 0.95$), and the degree of relative risk aversion is five ($\gamma = 5$).

When calibration results are reported below, all variables except for portfolio positions such as $a^i(s, s')$ and $\theta^i(s)$ are standardized by the total world endowment $e(z_t)$. Thus, what is implied by 'share' in this section is the ratio relative to the total endowment.

3.2 Purely transitory case

3.2.1 Almost perfect insurance outcomes in a time-0 setup

We first investigate how purely transitory catastrophic shocks are shared between two countries in the time-0 complete markets setup with solvency constraints. More concretely, a country-specific catastrophic shock reduces labor endowment by 20% with probability 1.8% per year, but without any persistence. That is, the labor endowment share of a damaged country declines from 0.5 to $\frac{1-0.2}{1+(1-0.2)}(1-\alpha)$ upon the realization of catastrophic shocks. When α is close to zero, the labor endowment share declines by about 5.6% $(= 0.5 - \frac{1-0.2}{1+(1-0.2)})$ due to a catastrophic event.

One of the most important observations about this case is that catastrophic shocks are insured almost perfectly between the two countries in spite of extremely low α $(= 0.1\%)$. Figure 1 plots the consumption share between a nondamaged country (country 1) and a damaged country (country 2); a catastrophic state takes place only in time 0. As demonstrated by Figure 1, the consumption share of the damaged country declines only by about 0.2% in time 0, although his labor endowment share declines by 5.6%. It implies that the damaged country can cover 5.4% out of 5.6% losses immediately after a catastrophic event. Even in a long term, the damaged country suffers from only 0.1% permanent losses unless another catastrophic shock hits this country.

In terms of asset pricing implications, when α is 0.1%, the average equity premium (0.994%) is much closer to the perfect insurance premium that emerges when solvency constraints are absent (0.937%) than to the closed economy premium that emerges when cross-border risk-sharing is absent (4.981%).

A major reason for the almost perfect insurance outcome in a time-0 setup is that Lucas trees whose dividends are proportional to the total endowment can serve as an effective insurance instrument.

3.2.2 Recovering the time-0 equilibrium outcome in a sequential setup

However, it is impossible to recover the above time-0 equilibrium outcome in a sequential setup, because a large-scale net flow of funds from country 1 to country 2 takes place at a catastrophic event in a time-0 setup. As described below, collateral constraints are so severely binding as to prevent the time-0 constrained efficient outcome from being realized in a sequential setup. Note that all asset prices are standardized by the total endowment.

The solvency of country 1 as an insurer is crucially limited as follows. As equation (12) implies, country 1 (insurer) can offer insurance payments to country 2 (insured) only up to $(p(s^0) + \alpha)\theta^1(s^{-1})$ where $\theta^1(s^{-1}) = 0.5$ in time 0. Given the time-0 equilibrium asset pricing, $(p(s^0) + \alpha) \times 0.5$ amounts to only 0.7% of the total world endowment. Accordingly, the catastrophe insurance payment from country 1 to country 2 (0.7%) is far short of the catastrophic loss borne by country 2 (5.6%). In other words, one-period insurance contracts can cover only a part of the realized losses of country 2. It thus follows that country 2 needs to borrow resources from country 1 in order to achieve the time-0 equilibrium outcome in a sequential setup, but that as equation (16) implies, it cannot at all.

From time 1 on, on the other hand, there never emerges a positive net flow of funds from one country to the other in the time-0 equilibrium outcome. Thus, except for a catastrophic event (time 0), the time-0 equilibrium outcome can be recovered even in a sequential setup.

A case with over-evaluation of Lucas trees As suggested in the previous section, we introduce the random pricing error associated with Lucas trees at a catastrophic event in order to relax the extent that collateral constraints are binding in a sequential setup. It is assumed that a pricing error (ϵ) takes *positive* 10% deviation from arbitrage pricing with probability one half and *negative* 10% deviation with probability one half, when a catastrophic event takes place in time 0.

Table 1 reports the case with positive 10% deviation. In this case, Lucas trees ($\theta^i(s^0)$) are rich relative to one-period contingent bonds, which consist of one-period contingent claims ($a^i(s^0, s^1)$). Thus, a damaged country (country 2) can exploit arbitrage profits and thereby cover uninsured catastrophic losses by making short positions in Lucas trees and long positions in contingent bonds. On the other hand, a nondamaged country can construct investment opportunities to smooth temporary relative gains over time by making long positions in Lucas trees and short positions in contingent bonds.

More concretely, with 10% over-evaluation of Lucas trees in time 0 at a catastrophic event, a damaged country receives 0.8% insurance payment from a nondamaged country and 0.8% gross returns from its own investment in Lucas trees. Consequently, the uncovered catastrophic loss borne by country 2 amounts to 4.0% ($5.6\% - 0.8\% - 0.8\%$). Then, a damaged country can finance uncovered losses up to 3.8% (out of 4.0%) by making 67.5% short positions in Lucas trees ($-\widehat{p}(s^0)\theta^2(s^0)$), and 63.7% long positions in contingent claims ($\sum_{s^1 \succeq s^0} q(s^0, s^1)a^2(s^0, s^1)$). Conversely, a nondamaged country can create investment opportunities by making 69.0% long positions in Lucas trees ($\widehat{p}(s^0)\theta^1(s^0)$), and 63.7% short positions in contingent bonds ($-\sum_{s^1 \succeq s^0} q(s^0, s^1)a^1(s^0, s^1)$).

Let us take a look at the time-1 payoff structure of Lucas trees and one-period contingent bonds from the perspective of a damaged country (country 2). Then, we can show that Lucas trees serve as relatively risky bonds, while contingent bonds play a role as relatively safe bonds.

As shown in Table 2, country 2's time-1 receipt from long positions in contingent bonds ($a^2(s^0, s^1)$) is 96.8% of the total world endowment in a case where no catastrophic shock is realized in either country, 67.5% in a case where a catastrophic shock hits on country 2, 66.0% in a case where a catastrophic shock hits on country 1, 42.3% in a case where catastrophic shocks hit on both countries. On the other hand, country 2's time-1 payment on short positions in Lucas trees ($-(p(s^1) + \alpha)\theta^2(s^0)$) is 96.8%, 66.0%, 66.0%, and 42.3% respectively. Then, contingent bonds are safe relative to Lucas trees by the insurance effect when a catastrophic event hits on country 2 in time 1; in this state, country 2 receives a positive net payoff ($+1.5\% = 67.5\% - 66.0\%$).

In sum, when Lucas trees are over-evaluated at a catastrophic event, a damaged country can effectively finance uncovered losses by making short positions in rich risky bonds (Lucas trees in this case) and long positions in cheap safe bonds (one-period contingent bonds in this case), while a nondamaged country can construct effective investment opportunities by making the opposite financial positions. From the viewpoint of a damaged country, safe bonds (one-period contingent bonds) serve as collateral assets in issuing risky bonds (Lucas trees).

A case with under-evaluation of Lucas trees Table 3 reports the case with negative 10% deviation. In this case, Lucas trees are cheap relative to one-period contingent bonds. Thus, a damaged country (country 2) can cover uninsured catastrophic losses by making long positions in Lucas trees and short positions in contingent bonds. On the other hand, by standing on the opposite side of a damaged country, a nondamaged country can construct investment opportunities to smooth temporary relative gains over time.

More concretely, with 10% under-evaluation of Lucas trees in time 0 at a catastrophic event, a damaged country receives 0.7% insurance payment from a nondamaged country and 0.7% gross returns from its own investment in Lucas trees. Consequently, the uncovered catastrophic loss borne by country 2 amounts to 4.2% (5.6% − 0.7% − 0.7%). Then, a damaged country can finance uncovered losses up to 4.0% (out of 4.2%) by making 26.9% long positions in Lucas trees ($\widehat{p}(s^0)\theta^2(s^0)$), and 31.0% short positions in contingent claims ($-\sum_{s^1 \succeq s^0} q(s^0, s^1)a^2(s^0, s^1)$). Conversely, a nondamaged country can create investment opportunities by making 25.7% short positions in Lucas trees ($-\widehat{p}(s^0)\theta^1(s^0)$), and 31.0% long positions in contingent bonds ($\sum_{s^1 \succeq s^0} q(s^0, s^1)a^1(s^0, s^1)$).

In the above case, Lucas trees serve as relatively safe bonds, while one-period contingent bonds play a role as relatively risky bonds. As shown in Table 4, country 2′s time-1 payment on short positions in contingent bonds ($-a^2(s^0, s^1)$) is 47.2% of the total world endowment in a case where no catastrophic shock is realized in either country, 30.7% in a case where a catastrophic shock hits on country 2, 32.2% in a case where a catastrophic shock hits on country 1, 20.6% in a case where catastrophic shocks hit on both countries. On the other hand, country 2′s time-1 receipt from long positions in Lucas trees ($(p(s^1) + \alpha)\theta^2(s^0)$) is 47.2%, 32.2%, 32.2%, and 20.6% respectively.

This time Lucas trees bonds are safe relative to contingent bonds by the insurance effect when a catastrophic event hits on country 2 in time 1; in this state, country 2 receives a positive net payoff (+1.5% = 32.2% − 30.7%).

In sum, when Lucas trees are under-evaluated at a catastrophic event, a damaged country can effectively finance uncovered losses by making short positions in rich risky bonds (one-period contingent bonds in this case) and long positions in cheap safe bonds (Lucas trees in this case), while a nondamaged country can construct effective investment opportunities by making the opposite financial positions. Again, from the viewpoint of a damaged country, safe bonds (Lucas trees) serve as collateral assets in issuing risky bonds (one-period contingent bonds).

Long-run effects on financial portfolio positions There never emerges any net flow of funds from one country to the other on time 1 onward even in the time-0 equilibrium outcome. Therefore, it is not necessary to introduce any random pricing error. As reported in Tables 1 and 3, the large-scale short and long positions that are build up at a catastrophic event scale down substantially in both countries from time 1 on. In the over-evaluation case, for example, country 2's short positions in Lucas trees reduce drastically, while in the under-evaluation case, country 2's long position in Lucas trees downsize greatly.

In any case, the identical portfolio appears from time 1 onward, unless another catastrophic event occurs. There, a nondamaged country continues to hold most of physical Lucas trees; that is, $\theta(s^t) \approx 1$ in the long run.

3.3 Some interpretations according to Euler inequalities

By taking a look at how Euler inequalities behave in a sequential setup, it may be easy to understand why the over- or under-evaluation of Lucas trees would help to recover the time-0 equilibrium outcome in a sequential setup. In the time-0 equilibrium outcome, Euler equations with respect to pricing of contingent claims hold for a solvency-unconstrained country, but they do not for a solvency-constrained country (see equation (11)). Hence, the following Euler inequality with respect to pricing of Lucas trees emerge if a country is subject to solvency constraints for at least one one-period ahead future state in a time-0 complete markets setup:

$$\sum_{s^1 \succeq s^0} \left[\beta \pi(s_1 | s_0) \frac{u'\left[c^i(s^{t+1})\right]}{u'\left[c^i(s^t)\right]} \frac{q(s^t, s^{t+1})\left[p(z^{t+1}) + d(z_{t+1})\right]}{p(z^t)} \right] \leq 1.$$

Then, how do Euler inequalities look like in a sequential setup with collateral constraints? Without any intervention in asset pricing, the time-0 equilibrium consumption profile between time 0 and time 1 ($c^i(s^0)$ and $c^i(s^1)$) cannot be achieved due to severe collateral constraints in a sequential setup, and the resulting profile between time 0 and time 1 ($\widehat{c}^i(s^0)$ and $\widehat{c}^i(s^1)$) deviates substantially from the time-0 equilibrium profile.

Suppose that a damaged country (country 2) cannot borrow resources from country 1 by making short positions in Lucas trees at a catastrophic event (time 0). Then, given the time-0 equilibrium asset pricing, the following Euler inequality holds for a damaged country:

$$\sum_{s^1 \succeq s^0} \left[\beta\pi(s_1|s_0) \frac{u'\left[\widehat{c}^2(s^1)\right]}{u'\left[\widehat{c}^2(s^0)\right]} \frac{q(s^0,s^1)\left[p(z^1)+d(z_1)\right]}{p(z^0)} \right] > 1.$$

The above Euler inequality implies that the time-0 equilibrium return on Lucas trees is too high for country 2 to borrow from country 1.

Suppose that a nondamaged country (country 1) cannot construct investment opportunities by making long positions in Lucas trees at a catastrophic event. Then, given the time-0 equilibrium asset pricing, the following Euler inequality holds for a nondamaged country:

$$\sum_{s^1 \succeq s^0} \left[\beta\pi(s_1|s_0) \frac{u'\left[\widehat{c}^1(s^1)\right]}{u'\left[\widehat{c}^1(s^0)\right]} \frac{q(s^0,s^1)\left[p(z^1)+d(z_1)\right]}{p(z^0)} \right] > 1.$$

The above Euler inequality implies that the consumption of country 1 grows too little due to missing investment opportunities, and that the resulting marginal rate of intertemporal substitution is too high for country 1. In the above case, the over-evaluation of Lucas trees would help to mitigate the extent that Euler equations deviate from one.

In the opposite case where a damaged country cannot borrow resources by making short positions in one-period contingent bonds, we have the following Euler inequalities:

$$\sum_{s^1 \succeq s^0} \left[\beta\pi(s_1|s_0) \frac{u'\left[\widehat{c}^2(s^1)\right]}{u'\left[\widehat{c}^2(s^0)\right]} \frac{q(s^0,s^1)\left[p(z^1)+d(z_1)\right]}{p(z^0)} \right] < 1,$$
$$\sum_{s^1 \succeq s^0} \left[\beta\pi(s_1|s_0) \frac{u'\left[\widehat{c}^1(s^1)\right]}{u'\left[\widehat{c}^1(s^0)\right]} \frac{q(s^0,s^1)\left[p(z^1)+d(z_1)\right]}{p(z^0)} \right] < 1.$$

Then, the under-evaluation of Lucas trees would help to reduce the deviation of Euler equations from one.

4 Conclusion

In this paper, we explore whether the efficient outcome achieved by a time-0 setup with solvency constraints can be recovered in a sequential setup with collateral constraints. It is in general impossible to recover the time-0 constrained efficient outcome in a sequential setup with collateral constraints, which are much more stringent than solvency constraints. However, when pricing errors occur randomly in pricing Lucas trees during a catastrophic event, it is possible to recover the time-0 solvency-constrained efficient outcome in a sequential manner.

More concretely, depending on the sign of pricing errors, either Lucas trees or contingent bonds are relatively risky in equilibrium, and there emerge richness in risky bonds and cheapness in safe bonds. Then, without violating collateral constraints, a damaged country can exploit arbitrage profits by making short positions in rich risky bonds and long positions in cheap safe bonds. From the viewpoint of a damaged country, safe bonds serve as collateral assets in issuing risky bonds. Even a nondamaged country benefits from the above financial transactions with damaged country, because it can obtain an investment opportunity to smooth temporary relative gains over time.

Thus, with a proper interpretation of random pricing errors associated with Lucas trees, the time-0 solvency-constrained efficient outcome may be interpreted as a realistic sequential equilibrium with collateral constraints even when solvency constraints are severely binding. Then, how can we interpret realistically such random pricing errors? Which kind of market interventions may correspond to the above stochastic components? As discussed so far, which assets are relatively risky is determined endogenously, and risky bonds are rich relative to safe bonds. Therefore, if some agent is expected to purchase risky bonds, which emerge endogenously, above arbitrage pricing during a catastrophic event, then the time-0 equilibrium outcome may be achieved in a sequential setup.

One possible candidate for such an agent may be a central bank. A central bank is indeed expected to intervene heavily in risky bond markets during a financial crisis. Along some implications implied by our theoretical exercise, such active interventions in financial markets would trigger the creation of good collateral when it is urgently needed in a crisis. With the above intervention, the resulting resource allocation could be more efficient.

Appendix: The numerical computation methods

As mentioned in Section 2, it is not possible to directly solve the sequential trading problem characterized by equation (14) because of the presence of solvency constraints. Following Lustig (2007), we instead solve the time-0 cost minimization problem dual to the utility maximization problem. We omit the time subscript t because the problem is formulated in a recursive manner.

We below standardize all endogenous variables except for asset volume by the total world endowment.[7] Accordingly, we transform the stochastic discount factors as follows:

$$\hat{\pi}\left(s'|s\right) = \frac{\pi(s'|s)\left(\frac{e(z')}{e(z)}\right)^{1-\gamma}}{\sum_{s'\in S}\pi(s'|s)\left(\frac{e(z')}{e(z)}\right)^{1-\gamma}}, \tag{19}$$

$$\hat{\beta}\left(s\right) = \beta\sum_{s'\in S}\pi(s'|s)\left(\frac{e(z')}{e(z)}\right)^{1-\gamma}. \tag{20}$$

We can use as a state variable the stationary consumption share ($\omega^i\left(s^t\right) = \frac{\zeta^i(s^t)^{\frac{1}{\gamma}}}{h(s^t)} \in [0,1]$) instead of the Negishi weight $\zeta^i(s^t)$. As the Negishi weight is revised upward upon default, the consumption share is revised upward based on a cutoff rule as described below.

There are two steps in finding the equilibrium pricing and allocation. Given the initially guessed liquidity shocks $g^{\text{guess}}(s'|s)$, the first step consists of solving the cost minimization problem given the sequence of prices, and of deriving optimal policy functions. In the second step, the sequence of consumption and asset pricing is computed from the simulation based on the derived policy functions; it is possible to map from liquidity shocks $g(s'|s)$ to stochastic discount factors $\beta\left(\frac{e(z')}{e(z)}\right)^{-\gamma}g(s'|s)^{\gamma}$, and to compute equilibrium asset pricing. We repeat this two-step procedure until the initially guessed liquidity shocks $g^{\text{guess}}(s'|s)$ coincide with the newly generated liquidity shocks $g^{\text{new}}(s'|s)$.

In solving the cost minimization problem, the current history is replaced by a trun-

[7] Alvarez and Jermann (2000) adopt the same transformation to make endogenous variables stationary in the case where the total endowment is growing.

cated history z^k. Here, the control variable is not current consumption, but a consumption share ω^i in a detrended version of the cost function (5), and it is rewritten in a recursive manner:

$$\hat{C}(\omega^i(s), s, z^k) = \min_{\omega^i} \left[\omega^i + \hat{\beta}(s) \sum_{s' \in S} \hat{\pi}(s'|s) g(s'|s)^{\gamma} \hat{C}(\omega^{i\prime}, s', z^{k'}) \right], \tag{21}$$

where $\hat{\pi}(s'|s)$ and $\hat{\beta}(s)$ are defined in equations (19) and (20). Note that $\hat{\pi}(s'|s)g(s'|s)^{\gamma}$ in the cost function corresponds to a stochastic discount factor or a pricing kernel; as a result of detrending, $(\frac{e(z')}{e(z)})$ is always equal to one.

Similarly, a detrended version of the present value of the endowment sequence is written as follows:

$$\hat{C}^e(s, z^k) = \hat{e}^i(s) + \hat{\beta}(s) \sum_{s' \in S} \hat{\pi}(s'|s) g(s'|s)^{\gamma} \hat{C}^e(s', z^{h'}),$$

where $\hat{e}^i(s)$ is the share of individual labor endowment to the aggregate endowment. Because ω^i is bounded from below upon default, the lower bound of ω^i or $\underline{\omega}(s)$ is determined by:

$$\hat{C}(\underline{\omega}(s), s, z^k) = \hat{C}^e(s, z^k).$$

Lustig (2007) finds that $\omega^i(s')$ is bounded from $\underline{\omega}(s')$ as a result of binding solvency constraints, and constructs the following cutoff rule to revise a state variable ω^i upward: that is, if $\omega^i(s) > \underline{\omega}(s')$, then $\omega^i(s') = \frac{\omega^i(s)}{g(s'|s)}$, and if $\omega^i(s) \leq \underline{\omega}(s')$, then $\omega^i(s') = \frac{\underline{\omega}(s')}{g(s'|s)}$. Given the exogenous endowment process (s, z^k), the social planner solves the above cost minimization problem together with the cutoff rule by adjusting the current consumption share $\omega^i(s)$ and the share allowed in the next period $\omega^i(s')$.

Because equation (21) is a standard dynamic programming problem, we can solve it by a policy function iteration procedure. For this purpose, the cost function is approximated by a cubic spline interpolation with 100 grids for state $\omega^i \in [0, 1]$. In addition, it is assumed that $k = 3$ for the history parameter. It is possible to derive a policy function $\omega' = f(\omega, s, z^k)$ from the computed cost function. It is also possible to obtain the share of consumption of the two countries from the sequence of promised consumption shares ω^i.

In simulation, we first generate the sequence of aggregate and idiosyncratic shocks $\{s_t\}_{t=1}^{31,000}$ for 31,000 periods while omitting the initial 1000 periods. Given this generated sequence, we derive the sequence of consumption shares from the computed policy function, and then compute asset pricing and liquidity shocks. As mentioned above, we repeat this procedure until the generated liquidity shocks converge.

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Table 1: Portfolio transaction behavior with one-time shock (shock size: 20%, α=0.1%, +10% deviation from arbitrage equity prices)

	labor endowment share	realized tree value $(p(s)+\alpha)\theta^i(s^{-1})$	insurance receipt $a^i(s^{-1},s)$	consumption share	invested trees $\theta^i(s)$	invested trees value $p(s)\theta^i(s)$	invested contingent claims $\sum q(s,s')a^i(s,s')$
			nondamaged country				
time -1	0.500	0.011	0.000	0.500	0.500	0.010	0.000
time 0	0.556	0.008	-0.008	0.502	46.360	0.690	-0.637
time 1	0.500	0.989	-0.968	0.502	1.620	0.033	-0.014
time 2	0.500	0.034	-0.013	0.501	1.290	0.026	-0.007
time 3	0.500	0.027	-0.006	0.501	1.130	0.023	-0.004
time 4	0.500	0.024	-0.003	0.501	1.050	0.021	-0.002
time 5	0.500	0.022	-0.001	0.501	1.010	0.020	-0.001
time 6	0.500	0.021	0.000	0.501	0.990	0.020	-0.001
time 7	0.500	0.021	0.000	0.501	0.980	0.020	0.000
time 8	0.500	0.021	0.000	0.501	0.980	0.020	0.000
time 9	0.500	0.021	0.000	0.501	0.970	0.020	0.000
time 10	0.500	0.021	0.001	0.501	0.970	0.020	0.000
time 11	0.500	0.021	0.001	0.501	0.970	0.020	0.000
			damaged country				
time -1	0.500	0.011	0.000	0.500	0.500	0.010	0.000
time 0	0.444	0.008	0.008	0.498	-45.360	-0.675	0.637
time 1	0.500	-0.968	0.968	0.498	-0.620	-0.013	0.014
time 2	0.500	-0.013	0.013	0.499	-0.290	-0.006	0.007
time 3	0.500	-0.006	0.006	0.499	-0.130	-0.003	0.004
time 4	0.500	-0.003	0.003	0.499	-0.050	-0.001	0.002
time 5	0.500	-0.001	0.001	0.499	-0.010	0.000	0.001
time 6	0.500	0.000	0.000	0.499	0.010	0.000	0.001
time 7	0.500	0.000	0.000	0.499	0.020	0.000	0.000
time 8	0.500	0.000	0.000	0.499	0.020	0.000	0.000
time 9	0.500	0.000	0.000	0.499	0.030	0.001	0.000
time 10	0.500	0.001	-0.001	0.499	0.030	0.001	0.000
time 11	0.500	0.001	-0.001	0.499	0.030	0.001	0.000

Note: All variables except for the number of labor endowment share and invested Lucas trees (θ^i) represent the ratio relative to the total world endowment. The labor endowment share in the second column represents the ratio relative to the total labor endowment.

Table 2: Receipts from or repayments on contingent contracts at maturity with one-time shock (shock size: 20%, α=0.1%, +10% deviation from arbitrage equity prices)

	shares of invested trees $\theta^i(s^{-1})$	no catastrophic shock realized $a^i(s^{-1}, s)$	catastrophic shock on country 1 $a^i(s^{-1}, s)$	catastrophic shock on country 2 $a^i(s^{-1}, s)$	catastrophic shock on country 1 and 2 $a^i(s^{-1}, s)$
			nondamaged country		
time 0	0.500	0.000	0.007	-0.007	0.000
time 1	46.360	-0.968	-0.660	-0.675	-0.423
time 2	1.620	-0.013	-0.009	-0.024	-0.006
time 3	1.290	-0.006	-0.004	-0.019	-0.003
time 4	1.130	-0.003	-0.002	-0.016	-0.001
time 5	1.050	-0.001	-0.001	-0.015	0.000
time 6	1.010	0.000	0.000	-0.015	0.000
time 7	0.990	0.000	0.000	-0.014	0.000
time 8	0.980	0.000	0.000	-0.014	0.000
time 9	0.980	0.000	0.000	-0.014	0.000
time 10	0.970	0.001	0.000	-0.014	0.000
time 11	0.970	0.001	0.000	-0.014	0.000
time 12	0.970	0.001	0.000	-0.014	0.000
			damaged country		
time 0	0.500	0.000	-0.007	0.007	0.000
time 1	-45.360	0.968	0.660	0.675	0.423
time 2	-0.620	0.013	0.009	0.024	0.006
time 3	-0.290	0.006	0.004	0.019	0.003
time 4	-0.130	0.003	0.002	0.016	0.001
time 5	-0.050	0.001	0.001	0.015	0.000
time 6	-0.010	0.000	0.000	0.015	0.000
time 7	0.010	0.000	0.000	0.014	0.000
time 8	0.020	0.000	0.000	0.014	0.000
time 9	0.020	0.000	0.000	0.014	0.000
time 10	0.030	-0.001	0.000	0.014	0.000
time 11	0.030	-0.001	0.000	0.014	0.000
time 12	0.030	-0.001	0.000	0.014	0.000

Note: $a^i(s^{-1}, s)$ is standardized by the total world endowment.

Table 3: Portfolio transaction behavior with one-time shock (shock size: 20%, α=0.1%, -10% deviation from arbitrage equity prices)

	labor endowment share	realized tree value $(p(s)+\alpha)\theta^i(s^{-1})$	insurance receipt $a^i(s^{-1},s)$	consumption share	invested trees $\theta^i(s)$	invested trees value $p(s)\theta^i(s)$	invested contingent claims $\sum q(s,s')a^i(s,s')$
			nondamaged country				
time -1	0.500	0.011	0.000	0.500	0.500	0.010	0.000
time 0	0.556	0.007	-0.007	0.502	-21.100	-0.257	0.310
time 1	0.500	-0.450	0.472	0.502	1.620	0.033	-0.014
time 2	0.500	0.034	-0.013	0.501	1.290	0.026	-0.007
time 3	0.500	0.027	-0.006	0.501	1.130	0.023	-0.004
time 4	0.500	0.024	-0.003	0.501	1.050	0.021	-0.002
time 5	0.500	0.022	-0.001	0.501	1.010	0.020	-0.001
time 6	0.500	0.021	0.000	0.501	0.990	0.020	-0.001
time 7	0.500	0.021	0.000	0.501	0.980	0.020	0.000
time 8	0.500	0.021	0.000	0.501	0.980	0.020	0.000
time 9	0.500	0.021	0.000	0.501	0.970	0.020	0.000
time 10	0.500	0.021	0.001	0.501	0.970	0.020	0.000
time 11	0.500	0.021	0.001	0.501	0.970	0.020	0.000
			damaged country				
time -1	0.500	0.011	0.000	0.500	0.500	0.010	0.000
time 0	0.444	0.007	0.007	0.498	22.100	0.269	-0.310
time 1	0.500	-0.968	0.968	0.498	-0.620	-0.013	0.014
time 2	0.500	-0.013	0.013	0.499	-0.290	-0.006	0.007
time 3	0.500	-0.006	0.006	0.499	-0.130	-0.003	0.004
time 4	0.500	-0.003	0.003	0.499	-0.050	-0.001	0.002
time 5	0.500	-0.001	0.001	0.499	-0.010	0.000	0.001
time 6	0.500	0.000	0.000	0.499	0.010	0.000	0.001
time 7	0.500	0.000	0.000	0.499	0.020	0.000	0.000
time 8	0.500	0.000	0.000	0.499	0.020	0.000	0.000
time 9	0.500	0.000	0.000	0.499	0.030	0.001	0.000
time 10	0.500	0.001	-0.001	0.499	0.030	0.001	0.000
time 11	0.500	0.001	-0.001	0.499	0.030	0.001	0.000

Note: All variables except for the number of labor endowment share and invested Lucas trees (θ^i) represent the ratio relative to the total world endowment. The labor endowment share in the second column represents the ratio relative to the total labor endowment.

Table 4: Receipts from or repayments on contingent contracts at maturity with one-time shock (shock size: 20%, α=0.1%, -10% deviation from arbitrage equity prices)

	shares of invested trees $\theta^i(s^{-1})$	no catastrophic shock realized $a^i(s^{-1},s)$	catastrophic shock on country 1 $a^i(s^{-1},s)$	catastrophic shock on country 2 $a^i(s^{-1},s)$	catastrophic shock on country 1 and 2 $a^i(s^{-1},s)$
			nondamaged country		
time 0	0.500	0.000	0.007	-0.007	0.000
time 1	-21.100	0.472	0.322	0.307	0.206
time 2	1.620	-0.013	-0.009	-0.024	-0.006
time 3	1.290	-0.006	-0.004	-0.019	-0.003
time 4	1.130	-0.003	-0.002	-0.016	-0.001
time 5	1.050	-0.001	-0.001	-0.015	0.000
time 6	1.010	0.000	0.000	-0.015	0.000
time 7	0.990	0.000	0.000	-0.014	0.000
time 8	0.980	0.000	0.000	-0.014	0.000
time 9	0.980	0.000	0.000	-0.014	0.000
time 10	0.970	0.001	0.000	-0.014	0.000
time 11	0.970	0.001	0.000	-0.014	0.000
time 12	0.970	0.001	0.000	-0.014	0.000
			damaged country		
time 0	0.500	0.000	-0.007	0.007	0.000
time 1	22.100	-0.472	-0.322	-0.307	-0.206
time 2	-0.620	0.013	0.009	0.024	0.006
time 3	-0.290	0.006	0.004	0.019	0.003
time 4	-0.130	0.003	0.002	0.016	0.001
time 5	-0.050	0.001	0.001	0.015	0.000
time 6	-0.010	0.000	0.000	0.015	0.000
time 7	0.010	0.000	0.000	0.014	0.000
time 8	0.020	0.000	0.000	0.014	0.000
time 9	0.020	0.000	0.000	0.014	0.000
time 10	0.030	-0.001	0.000	0.014	0.000
time 11	0.030	-0.001	0.000	0.014	0.000
time 11	0.030	-0.001	0.000	0.014	0.000

Note: $a^i(s^{-1},s)$ is standardized by the total world endowment.

Figure 1: Consumption shares with one-time shock (shock size: 20%, α=0.1%)